TOPOLOGICAL INVARIANTS OF O'GRADY'S SIX DIMENSIONAL IRREDUCIBLE SYMPLECTIC VARIETY

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ABSTRACT. We study O'Grady examples of irreducible symplectic varieties: we establish that both of them can be deformed into lagrangian fibrations. We analyse in detail the topology of the six dimensional example: in particular we compute its Euler characteristic and determine its Beauville form.

INTRODUCTION

Irreducible symplectic varieties are defined as compact Kähler varieties having trivial fundamental group and endowed with a unique global holomorphic 2-form which is non degenerate on each point.

By the Bogomolov decomposition [Bo 74], irreducible symplectic varieties play (together with Calabi Yau manifolds and Complex tori) a central role in the classification of compact Kähler manifolds with torsion c_1.

Very few examples of irreducible symplectic varieties are available in literature.

For any positive integer n Beauville exhibited 2 examples of dimension $2n$ ([Be 83]): the Hilbert scheme $Hilb^n(X)$ parametrizing 0 dimensional subschemes of length n on a K3 surface X, and the Kummer generalized variety $K^n(T)$ of a 2-dimensional torus T, namely the locus in $Hilb^{n+1}(T)$ parametrizing subschemes having associated cycle summing up to zero.

Besides the Beauville examples, there are only two known examples of irreducible symplectic varieties up to deformation equivalence: they have been exhibited by O'Grady in [OG 99] and [OG 03] and their dimensions are respectively ten and six.

While Hilbert schemes of points and Kummer generalized varieties have been deeply studied, very little is known about the two O'Grady examples.

In this paper, after having proved that both the O'Grady examples can be deformed into Lagrangian fibrations (see Corollary 1.1.15) we study the topology of the one, $\widetilde{\mathcal{M}}$, having dimension six. In section 2 we establish that its Euler characteristic is 1920 (see Theorem 2.2.3). Finally, in section 3 we determine the Beauville form and the Fujiki constant of $\widetilde{\mathcal{M}}$ (see Theorem 3.5.1).

It is remarkable that the Fujiki constant of $\widetilde{\mathcal{M}}$ is 60, as in the case of the generalized Kummer variety of the same dimension: this is the first known case where two non diffeomorphic irreducible symplectic varieties have the same Fujiki constant.

1. O'GRADY'S DESINGULARIZATION

In this section we recall the construction of O'Grady's examples of irreducible symplectic varieties and we give a slight generalization of O'Grady's symplectic desingularization (see propositions 1.1.8 and 1.1.11). This desingularization enables us to prove in Corollary 1.1.15

that all known examples of irreducible symplectic varieties are deformation equivalent to lagrangian fibrations. In the rest of the section we fix the notation that we will use in the next 2 sections where we study the six dimensional O'Grady example $\widetilde{\mathcal{M}}$.

1.1. **Analysis of O'Grady's construction.** We need to recall two general definitions due to Mukai. In these definitions X is a projective symplectic surface, namely X is a $K3$ or an abelian surface.

Definition 1.1.1. Consider the involution on the even cohomology of X given by

$$
\begin{array}{ccc}
H^{ev}(X, \mathbb{Z}) & \longrightarrow & H^{ev}(X, \mathbb{Z}) \\
\alpha = \alpha_0 + \alpha_2 + \alpha_4 & \mapsto & \overline{\alpha} = \alpha_0 - \alpha_2 + \alpha_4
\end{array}
$$

with $\alpha_{2i} \in H^{2i}(X, \mathbb{Z})$. Let $\alpha, \beta \in H^{ev}(X, \mathbb{Z})$: the Mukai's pairing $< \cdot, \cdot >$ is given by:

$$
< \alpha, \beta > := - \int \overline{\alpha} \cup \beta.
$$

Finally the Mukai lattice is $(H^{ev}(X, \mathbb{Z}), < \cdot, \cdot >)$.

Definition 1.1.2. Let F be a coherent sheaf on X, the Mukai vector of F is

$$
m(F) := ch(F) \cup \sqrt{Td(X)} \in H^{ev}(X, \mathbb{Z}).
$$

Notation 1.1.3. Letting $\eta \in H^4(X, \mathbb{Z})$ be the fundamental class, given

$$
v = (v_0, v_2, v_4\eta) \in H^0(X, \mathbb{Z}) \oplus \mathbf{Pic}(X) \oplus H^4(X, \mathbb{Z}),
$$

we will denote by $\mathbf{M}_v$ the Simpson moduli space of semistable sheaves on X having as Mukai's vector the class of v in $H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$.
Using the identification $H^4(X, \mathbb{Z}) \equiv \mathbb{Z}$ we will usually denote $v = (v_0, v_2, v_4\eta)$ by (v_0, v_2, v_4). Moreover, to simplify the notation, we will always replace v_2 by a divisor in the linear equivalence class defined by v_2.

In this section we simply fix the notation and briefly recall the construction of O'Grady's examples of irreducible symplectic varieties.
In all this paper $\mathcal{J}$ will be the Jacobian of a genus 2 curve C^0 such that $\mathbf{NS}(\mathcal{J}) = \mathbb{Z}c_1(\Theta)$ where Θ is a symmetric theta divisor .
According to Notation 1.1.3, given

$$
v = (v_0, v_2, v_4) \in H^0(\mathcal{J}, \mathbb{Z}) \oplus \mathbf{Pic}(\mathcal{J}) \oplus H^4(\mathcal{J}, \mathbb{Z}),
$$

we will denote by $\mathbf{M}_v$ the Simpson moduli space of semistable sheaves on $\mathcal{J}$ having as Mukai's vector the class of v in $H^{ev}(\mathcal{J}, \mathbb{Z})$.
The moduli space $\mathbf{M}_v$ is endowed with a regular morphism defined by

$$
(1) \qquad
\begin{array}{cccc}
a_v: & \mathbf{M}_v & \longrightarrow & \widehat{\mathcal{J}} \times \mathcal{J} \\
& [F] & \longmapsto & (Det(F), \sum c_2(F))
\end{array}
$$

where $[F]$ is the S-equivalence class of F, $Det(F)$ is the determinant bundle of F and, if the formal sum $\sum n_i p_i$ is a representative of the second Chern class of F in the Chow ring of $\mathcal{J}$, then $\sum c_2(F) := \sum n_i p_i \in \mathcal{J}$.
This enables us to give the following definition

Definition 1.1.4.

$$
\mathbf{M}_v^0 := a_v^{-1}(v_2, 0).
$$

Obviously the points of $\mathbf{M}_v^0$ parametrize S-equivalence classes of semistable sheaves having determinant bundle linearly equivalent to v_2 and second Chern class summing up to 0.

In the paper [OG 03] O'Grady constructed his second example of irreducible symplectic variety starting from the moduli space $\mathbf{M}_{(2,0,-2)}$.

The singular locus Σ of $\mathbf{M}_{(2,0,-2)}$ parametrizes polystable sheaves of the form

$$I_p \otimes L_1 \oplus I_q \otimes L_2$$

where p and q are points on $\mathcal{J}$, I_p and I_q are the corresponding sheaves of ideals, and L_i are line bundles with homologically trivial first Chern class.

The singular locus Ω of Σ is precisely the subscheme parametrizing sheaves

$$I_p \otimes L \oplus I_p \otimes L.$$

The first step in constructing the O'Grady 6-dimensional irreducible symplectic variety is the following: blow up $\mathbf{M}_{(2,0,-2)}$ along Ω and then blow up $Bl_\Omega\mathbf{M}_{(2,0,-2)}$ along the strict transform of Σ and finally contract the inverse image of Ω via the two blow ups. This produces a ten dimensional smooth variety $\widetilde{\mathbf{M}}_{(2,0,-2)}$. Moreover the obvious birational map from $\widetilde{\mathbf{M}}_{(2,0,-2)}$ to $\mathbf{M}_{(2,0,-2)}$, extends to a regular morphism

$$\widetilde{\pi} : \widetilde{\mathbf{M}}_{(2,0,-2)} \longrightarrow \mathbf{M}_{(2,0,-2)}.$$

Let $a_{(2,0,-2)}$ be defined by (1), and define

(2) $$\widetilde{a}_{(2,0,-2)} := a_{(2,0,-2)} \circ \widetilde{\pi} : \widetilde{\mathbf{M}}_{(2,0,-2)} \longrightarrow \widehat{\mathcal{J}} \times \mathcal{J}.$$

The new six dimensional irreducible symplectic variety of O'Grady is then given by:

Definition 1.1.5.
$$\widetilde{\mathcal{M}} := \widetilde{a}_{(2,0,-2)}^{-1}(0,0).$$

Remark 1.1.6. $\widetilde{\mathcal{M}}$ can be equivalently constructed starting from the locus $\mathbf{M}_{(2,0,-2)}^0 = a_{(2,0,-2)}^{-1}(0,0)$ blowing up $\mathbf{M}_{(2,0,-2)}^0$ along $\Omega \cap \mathbf{M}_{(2,0,-2)}^0$, blowing up here the strict transform of $\Sigma \cap \mathbf{M}_{(2,0,-2)}^0$ and contracting the inverse image via the composition of the two blow ups of $\Omega \cap \mathbf{M}_{(2,0,-2)}^0$.

Remark 1.1.7. There are Mukai vectors different from $(2,0,-2)$ for which O'Grady's construction in [OG 99], sketched above, works and produces a smooth algebraic variety with a holomorphic symplectic two form. They probably do not give new deformation classes of irreducible symplectic varieties, but at least two of them will be useful in this paper to understand the geometry of O'Grady's examples, so in the next two propositions we state which Mukai vectors admit such a weak generalization of O'Grady's result and in the successive corollary we single out the cases that we will effectively use later.

Proposition 1.1.8. *Let X be a K3 surface or an abelian surface such that $\mathbf{NS}(X) = \mathbb{Z}H$. Let $v \in H^0(X,\mathbb{Z}) \oplus \mathbf{Pic}(X) \oplus H^4(X,\mathbb{Z})$ and let $\overline{v} \in H^{ev}(X,\mathbb{Z})$ be the class of v.*
Suppose that:

    (1) $2|\overline{v}$, *but* $\frac{\overline{v}}{2}$ *is primitive,*
    (2) $< \overline{v}, \overline{v} >= 8$,
    (3) *The moduli space $\mathbf{M}_{\frac{v}{2}}$ is fine and non empty.*

Then $\mathbf{M}_v$ is reduced and there exists a symplectic desingularization

$$\widetilde{\pi}_v : \widetilde{\mathbf{M}}_v \to \mathbf{M}_v.$$

It can be obtained exactly repeating O'Grady's construction in [OG 99].

Remark 1.1.9. A slight modification of the following proof shows that Proposition 1.1.8 still holds if we only require in 3) $\mathbf{M}_{\frac{v}{2}} \neq \emptyset$.

This condition is always verified if the first non zero coefficient of $\frac{v}{2}$ is positive (see Theorem 0.1 of [Yo] and Theorem 0.1 of [Yo 01]).

Proof. The proof is exactly the same after replacing the sheaves of the form $I_p \otimes L$ by the ones whose Mukai vector is $\frac{1}{2}\overline{v}$. Since $\mathbf{NS}(X) = \mathbb{Z}H$, the Hilbert polynomial of a sheaf determines its Mukai vector, hence, by primitivity, $\mathbf{M}_{\frac{v}{2}}$ is smooth and a strictly semistable sheaf F such that $[F] \in \mathbf{M}_v$ fits in an exact sequence

$$(3) \qquad\qquad\qquad 0 \to G_1 \to F \to G_2 \to 0$$

with $[G_i] \in \mathbf{M}_{\frac{v}{2}}$.

For $\overline{v}$ divisible only by 2, the classification of the structures of semistable sheaves with their automorphism groups modulo scalars, is easily seen to be the following:

(1) $\frac{Aut(F)}{\mathbb{C}^*} = PGL(2)$ if $G_1 = G_2$ and the extension (3) is trivial,

(2) $\frac{Aut(F)}{\mathbb{C}^*} = (\mathbb{C}, +)$ if $G_1 = G_2$ and the extension (3) is non trivial,

(3) $\frac{Aut(F)}{\mathbb{C}^*} = \mathbb{C}^*$ if $G_1 \neq G_2$ and the extension (3) is trivial,

(4) $\frac{Aut(F)}{\mathbb{C}^*} = id$ if $G_1 \neq G_2$ and the extension (3) is non trivial.

In each of these items $[G_i] \in \mathbf{M}_{\frac{v}{2}}$: this generalizes Corollary (1.1.8) of [OG 99].

In section (1) of [OG 99] for any even $c \geq 4$ a desingularization $\widehat{\mathbf{M}}_{(2,0,2-c)}$ of $\mathbf{M}_{(2,0,2-c)}$ is constructed. Any statement proved in section (1) of [OG 99] has an obvious analogous if we replace the Mukai vector $(2,0,2-c)$ with a vector $\overline{v}$ satisfying our hypothesis. Furthermore, using the given classification of semistable sheaves, we can repeat exactly the same proofs if we assume the followings:

- $\mathbf{M}_{\frac{v}{2}}$ is endowed with a tautological family; this property of $\mathbf{M}_{(1,0,1-\frac{c}{2})}$ is used in the preparation and in the proof of Proposition (1.7.10),
- for $[G_1] \neq [G_2]$ in $\mathbf{M}_{\frac{v}{2}}$, there are non trivial extensions of G_1 by G_2; this property for $[G_i] \in \mathbf{M}_{(1,0,1-\frac{c}{2})}$ is used in the proof of Claim (1.4.8),
- $dim(Ext^1(G_i, G_i)) \geq 4$ for any $[G_i] \in \mathbf{M}_{\frac{v}{2}}$; this property for $G_i \in \mathbf{M}_{(1,0,1-\frac{c}{2})}$ is used in the proof of Lemma (1.5.6).

Since

$$dim(Ext^1(G_1, G_2)) = \{ \begin{array}{ll} \frac{<\overline{v},\overline{v}>}{4} & if\ G_1 \neq G_2 \\ \frac{<\overline{v},\overline{v}>}{4} + 2 & if\ G_1 = G_2 \end{array}$$

we get that under our hypothesis there exists a desingularization $\widehat{\mathbf{M}}_v$ of $\mathbf{M}_v$ obtained repeating formally O'Grady's desingularization of $\mathbf{M}_{(2,0,2-c)}$. In section (2) O'Grady fixes $c = 4$ in order to obtain a contraction of $\widehat{\mathbf{M}}_{(2,0,2-c)}$ to a symplectic variety $\widetilde{\mathbf{M}}_{(2,0,2-c)}$. For $c = 4$ the desingularization procedure is more simple: the simplification depends only on

$$dim(Ext^1(G, G)) = 4\ \forall G \in \mathbf{M}_{(1,0,1-\frac{c}{2})}$$

(see Formula (1.8.2) and the successive paragraph). In a completely analogous way, if

$$(4) \qquad\qquad dim(Ext^1(G, G)) = 4\ \forall G \in \mathbf{M}_{\frac{v}{2}}$$

the desingularization of $\mathbf{M}_v$ simply consists in blowing up $\mathbf{M}_v$ along the closed subvariety Ω_v whose points represent to sheaves of the form G^2 and then blowing up the strict transform of the closed subvariety Σ_v whose points represent all the semistable sheaves.

Since hypothesis 3) implies (4) and since any result in section 2) of [OG 99] is a consequence

of the analysis in section 1) we also get that those results still hold if we replace $\mathbf{M}_{(2,0,-2)}$ by $\mathbf{M}_v$ with v stasfying our hypotheses. In particular there exists a symplectic birational model $\widetilde{\mathcal{M}}_v$ of $\mathbf{M}_v$ and it is endowed with a regular map

$$\widetilde{\pi}_v : \widetilde{\mathbf{M}}_v \to \mathbf{M}_v$$

being an isomorphism on the smooth locus of $\mathbf{M}_v$.

It remains to prove that $\mathbf{M}_v$ is reduced. From section (1) of [OG 99] we can directly deduce that $\mathbf{M}_{(2,0,2-c)}$ is reduced. In fact $\mathbf{M}_{(2,0,2-c)}$ is the $PGL(N)$-quotient of the scheme $\mathcal{Q}_c$, so it will be enough to show that the semistable locus of $\mathcal{Q}_c$ is reduced. Since any point representing a semistable sheaf can be moved by $PGL(N)$ to any neighborhood of any point representing the polystable sheaf associated to it, it will be enough to prove that any point in $\mathcal{Q}_c$ representing a polystable sheaf has a reduced neighborhood. Neighborhoods in $\mathcal{Q}_c$ of points representing polystable sheaves are described in section (1) of [OG 99] . If the sheaf F represented by the point y is polystable, the $PGL(n)-$orbit of y is closed, and applying Luna's etale slice theorem (see pages 54-55 of [OG 99]) we get that a neighborhood of y has an etale covering from $PGL(n) \times_{st(y)} \mathcal{V}$, where $st(y)$ is the stabilizer of y and $\mathcal{V}$ is the etale slice. In the cases occurring in the study of $\mathcal{Q}_c$, we always get that $\mathcal{V}$ is reduced, locally irreducible near y. In fact, if $\mathcal{V}$ is not smooth near y (namely if F is not stable), the normal cone to $\mathcal{V}$ at y turns out to be either a the affine cone over a reduced irreducible quadric hypersurface if $F = G_1 \oplus G_2$ with $G_1 \neq G_2$ (see Claim (1.4.8) and Proposition (1.4.10)) or the affine cone over a reduced irreducible complete intersection of three quadrics (see Lemma (1.5.6) and Proposition (1.5.10)). It follows that $PGL(n) \times_{st(y)} \mathcal{V}$ is always reduced near $(1, y)$: hence its etale image in $\mathcal{Q}_c$ is reduced too.

Since as we said earlier any statement of section 1 of [OG 99] still holds, with the same proof, after replacing $(2, 0, 2 - c)$ with a v satisfying our hypotheses, then we get that for any such a v the moduli space $\mathbf{M}_v$ is reduced. $\square$

Remark 1.1.10. In the proof of the previous proposition we showed that all that is proved in sections 1) and 2) of [OG 03] holds under the hypothesis of the proposition. In particular we can describe the fiber of O'Grady's desingularization as follows. If $p \in \mathbf{M}_v$ corresponds to a sheaf F such that $\frac{Aut(F)}{\mathbb{C}^*} = PGL(2)$ then $\widetilde{\pi}_v^{-1}(p)$ is a smooth 3-dimensional quadric as in formula (2.2.9) on page 88 of [OG 99]. If $p \in \mathbf{M}_v$ corresponds to a sheaf F such that $\frac{Aut(F)}{\mathbb{C}^*} = \mathbb{C}^*$ then $\widetilde{\pi}_v^{-1}(p)$ is a $\mathbb{P}^1$ as shown in formula (2.2.4) on page 87 of [OG 99]. This will be used in the computation of the Euler characteristic of $\widetilde{\mathcal{M}}$.

Proposition 1.1.11. *Let $X = \mathcal{J}$, let v satisfy the hypothesis of Proposition 1.1.8. Let $\widetilde{\pi} : \widetilde{\mathbf{M}}_v \longrightarrow \mathbf{M}_v$ be the desingularization map obtained in the same proposition and let $\widetilde{\mathbf{M}}_v^0$ be as defined in 1.1.5, then $\widetilde{\mathbf{M}}_v^0$ is a smooth algebraic variety endowed with a symplectic holomorphic two form obtained restricting the one on $\widetilde{\mathbf{M}}_v$.*

Proof. We have only to prove that the symplectic form of $\widetilde{\mathbf{M}}_v$ restricts to a symplectic form on $\widetilde{\mathbf{M}}_v^0$: this proof can be copied from Proposition 1.1 of [De 99] and Proposition 2.3.3 of [OG 03]. $\square$

Remark 1.1.12. Notice that, when Proposition 1.1.8 and Proposition 1.1.11 actually work, namely when the moduli spaces involved are not empty, they always produce respectively pure 10 dimensional and pure 6 dimensional symplectic varieties.

Corollary 1.1.13. (1) *Set $X := \mathcal{J}$, the procedure described in the previous propositions produces a symplectic desingularization $\widetilde{\mathbf{M}}_{(0,2\Theta,-2)}^0$ of $\mathbf{M}_{(0,2\Theta,-2)}^0$.*

(2) *Let X be a K3 surface obtained as a double covering of the projective plane ramified along a smooth sextic. Suppose that $\mathbf{Pic}(X)$ is generated by H, the pull-back of a line. Then there exists a symplectic desingularization $\widetilde{\mathbf{M}}_{(0,2H,2)}$ of $\mathbf{M}_{(0,2H,2)}$.*

Proof. Conditions (1) and (2) of Proposition 1.1.8 are obviously satisfied, the condition (3) is satisfied using the criterion (see appendix of [Mu 84]) asserting that a tautological family for the stable locus of a moduli space $\mathbf{M}_v$ exists when $G.C.D(v_0, \{v_2 \cdot c_1(L)\}_{L \in \mathbf{Pic}(X)}, \chi) = 1$ (χ is the Euler characteristic of any sheaf of $\mathbf{M}_v$), indeed in the cases we are considering $\chi = -1$ or $\chi = 1$.   □

Remark 1.1.14. In Proposition 2.2.1 we will prove that $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ is birational to $\widetilde{\mathcal{M}}$, on the other hand in Proposition (4.1.5) of [OG 99] it is proved (with a different notation) that the 10 dimensional O'Grady's example is birational to an irreducible component N of $\mathbf{M}_{(0,2H,2)}$ (actually it can be proved that $\mathbf{M}_{(0,2H,2)}$ is irreducible). Since simple connectivity and $dim(H^{2,0})$ are birational invariants for smooth varieties, this implies that $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ and the irreducible component $\widetilde{N}$ (over N) of $\widetilde{\mathbf{M}}_{(0,2H,2)}$ are symplectic irreducible.

It is conjectured (see [Sa 03]) that any Irreducible symplectic variety is deformation equivalent to a Lagrangian fibration (i.e. a 2n dimensional symplectic variety endowed with a proper morphism to an n-dimensional variety, such that the restriction of the symplectic form to the general fiber is zero): in the following corollary we verify this conjecture on the known examples.

Corollary 1.1.15. *All the known irreducible symplectic varieties are deformation equivalent to lagrangian fibrations.*

Proof. It is well known for Hilbert schemes of points on a K3 surfaces (see [Be 99]) and for generalized Kummer varieties (see [De 99]). For the O'Grady examples, by the last remark and since birational irreducible symplectic varieties are deformation equivalent (see [Hu 97]), it is enough to prove that $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ and $\widetilde{N}$ are lagrangian fibrations. Both of these spaces are desingularizations of moduli spaces parametrizing sheaves on surfaces supported in codimension 1 : the functor associating to a sheaf the subscheme defined by its Fitting ideal (see 2.1.1)always defines, on any such moduli space, a regular map to the suitable Hilbert scheme. In the case of $N \subset \mathbf{M}^0_{(0,2H,2)}$ the Hilbert scheme is identified with the 5 dimensional linear system $|2H|$ and the morphism is easily seen (see 4.2 of [OG 99]) to be surjective.
In the case of $\mathbf{M}^0_{(0,2\Theta,-2)}$ we will see in section 3.2.4 that this regular morphism (we will denote this map by Φ) is surjective on a closed subvariety of the Hilbert scheme identified with the 3 dimensional linear system $|2\Theta|$.
Therefore both the irreducible symplectic varieties $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ and $\widetilde{N}$ have a surjective morphism to a projective space of dimension equal to half their dimension: by Theorem 1 of [Ma 01] these morphisms are lagrangian fibrations.   □

1.2. **Notation.** Now we fix the notations that we will follow for the rest of the paper.

Notation 1.2.1. Given a coherent sheaf F on our abelian surface $\mathcal{J}$, we denote by $[F]$ its S-equivalence class or equivalently the associated point in the moduli space.
Given $v \in H^0(\mathcal{J}, \mathbb{Z}) \oplus \mathbf{Pic}(\mathcal{J}) \oplus H^4(\mathcal{J}, \mathbb{Z})$ satisfying the hypotheses of Proposition 1.1.11, denoting by $\overline{v}$ its class in $H^{ev}(\mathcal{J}, \mathbb{Z})$, we set:
$\Sigma_v := \{[F_1 \oplus F_2] \in \mathbf{M}_v : m(F_1) = ch(F_1) = m(F_2) = ch(F_2) = \frac{1}{2}\overline{v}\}$ the singular locus of $\mathbf{M}_v$,
$\Omega_v := \{[F \oplus F] \in \mathbf{M}_v : m(F) = ch(F) = \frac{1}{2}\overline{v}\}$ the singular locus of Σ_v,
$\widetilde{\pi}_v : \widetilde{\mathbf{M}}_v \longrightarrow \mathbf{M}_v$ the symplectic desingularization map,

$\widetilde{\Sigma}_v := \widetilde{\pi}_v^{-1}\Sigma_v,$

$\widetilde{\Omega}_v := \widetilde{\pi}_v^{-1}\Omega_v,$

$a_v := Det \times \Sigma c_2 : \mathbf{M}_v \longrightarrow \mathbf{Pic}(\mathcal{J}) \times \mathcal{J},$

$\widetilde{a}_v := a_v \circ \widetilde{\pi}_v,$

$\widetilde{\mathbf{M}}_v^0 := (\widetilde{a}_v)^{-1}(v_2, 0),$

$\mathbf{M}_v^0 := (a_v)^{-1}(v_2, 0),$

$\widetilde{\pi}_v^0 : \widetilde{\mathbf{M}}_v^0 \longrightarrow \mathbf{M}_v^0$ the restriction of $\widetilde{\pi}_v,$

$\Sigma_v^0 := \mathbf{M}_v^0 \cap \Sigma_v,$

$\Omega_v^0 := \mathbf{M}_v^0 \cap \Omega_v,$

$\widetilde{\Sigma}_v^0 := \widetilde{\mathbf{M}}_v^0 \cap \widetilde{\Sigma}_v = \widetilde{\pi}_v^{-1}\Sigma_v^0,$

$\widetilde{\Omega}_v^0 := \widetilde{\mathbf{M}}_v^0 \cap \widetilde{\Omega}_v = \widetilde{\pi}_v^{-1}\Omega_v^0.$

When we will consider the case

(5) $$v = (2, 0, -2)$$

(namely the Mukai vector used by O'Grady) we will generally use the original notation $\widetilde{\mathcal{M}}$, $\widetilde{\Sigma}$ and $\widetilde{\pi}$ to denote $\widetilde{\mathbf{M}}_v^0$ and $\widetilde{\Sigma}_v^0$ and $\widetilde{\pi}_v^0$ respectively.

Remark 1.2.2. We will often implicitly use the identification $\mathcal{J} \simeq \widehat{\mathcal{J}}$ given by means of Θ. In particular the Fourier-Mukai transform $\mathcal{FM}$ will be seen as the self-equivalence of the derived category of coherent sheaves on $\mathcal{J}$ induced by the functor

$$F \mapsto q_{2*}(\mathcal{P} \otimes q_1^* F)$$

where $q_1 : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ is the projection on the ith-factor and $\mathcal{P} := q_1^*\mathcal{O}(\Theta) \otimes q_2^*\mathcal{O}(\Theta) \otimes m^*\mathcal{O}(\Theta)^\vee$ is the Poincaré line bundle ($m : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ is the sum). Analogously the map induced in cohomology by the Fourier-Mukai transform will be seen as an endomorphism of $H^{2\bullet}$: precisely the endomorphism given by $\alpha \mapsto q_{2*}(ch(\mathcal{P}) \otimes q_1^*\alpha)$.

Finally, given a sheaf F on $\mathcal{J}$ we will denote by $c_i(F)$ $(ch_i(F))$ the i-th Chern class (degree i component of the Chern character) of F in both the cohomology ring and the Chow ring: we will usually consider Chern classes (characters) as cohomology classes and in the opposite case we will explicitly say that we are referring to classes in the Chow ring.

2. Euler characteristic of $\widetilde{\mathcal{M}}$

In this section we prove (Theorem 2.2.3) that the Euler characteristic of $\widetilde{\mathcal{M}}$ is 1920. By Corollary 1.1.13 $\widetilde{\mathbf{M}}_{(0,2\Theta,-2)}^0$ is a symplectic desingularization of $\mathbf{M}_{(0,2\Theta,-2)}^0$. In subsection 1 we study $\mathbf{M}_{(0,2\Theta,-2)}^0$ and determine its Euler characteristic. Since we know the fibers of the desingularization morphism (see Remark 1.1.10), we can also deduce the Euler characteristic of $\widetilde{\mathbf{M}}_{(0,2\Theta,-2)}^0$ (see 2.1.7) .

In subsection 2 we give an explicit birational map between $\widetilde{\mathbf{M}}_{(0,2\Theta,-2)}^0$ and $\widetilde{\mathcal{M}}$. It turns out that $\widetilde{\mathbf{M}}_{(0,2\Theta,-2)}^0$ is an irreducible symplectic variety too. By a theorem due to Huybrechts the existence of a birational map implies that $\widetilde{\mathcal{M}}$ and $\widetilde{\mathbf{M}}_{(0,2\Theta,-2)}^0$ are deformation equivalent: hence their Euler characteristics are equal.

2.1. **Euler characteristic of $\widetilde{\mathbf{M}}_{(0,2\Theta,-2)}^0$.** The analysis of $\mathbf{M}_{(0,2\Theta,-2)}^0$ is simplified by the existence of the regular morphism

(6) $$\Phi : \mathbf{M}_{(0,2\Theta,-2)}^0 \to |2\Theta|$$

associating to an S-equivalence class of sheaves the fitting subscheme of each of its representative.

Remark 2.1.1. Recall that given a locally free presentation of a sheaf F,

$$F_1 \xrightarrow{f} F_0 \to F \to 0$$

the Fitting subscheme of F is defined as the cokernel of the map

$$\wedge^n F_1 \otimes \wedge^n F_0^\vee \to \mathcal{O}$$

(n being the rank of F_0) induced by f. In the case of pure 1-dimensional sheaves on a smooth surface the construction of the Fitting subscheme globalizes transforming flat families of sheaves into flat families of 1-dimensional subschemes: thus it induces regular maps between moduli spaces parametrizing pure 1-dimensional sheaves and Hilbert schemes parametrizing curves (see [LP 93]). Moreover we can actually choose, as a locally free presentation, a locally free resolution of F: this implies that the fundamental cycle of the Fitting subscheme of F is a representative of $c_1(F)$ in the Chow ring of $\mathcal{J}$, in particular, for $[F] \in \widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$, its Fitting subscheme belongs to $|2\Theta|$.

In order to study $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$, we now describe singularities occurring in curves in the linear system $|2\Theta|$.

Lemma 2.1.2. *Let* $C \in |2\Theta|$:
   (1) *If* $C \cap \mathcal{J}[2] = \emptyset$ *then either* C *is smooth or* $C = \Theta_x + \Theta_{-x}$ *and* $\Theta_x \cap \Theta_{-x}$ *consists of 2 distinct points*
   (2) *If* $p \in \mathcal{J}[2]$ *belongs to* C *then* C *is singular at* p: *in this case either* C *is an irreducible nodal curve smooth outside of* $\mathcal{J}[2]$ *or* $C = \Theta_x + \Theta_{-x}$ *with* $\Theta_x \cap \Theta_{-x} = p$ *or* $\Theta_x = \Theta_{-x}$

Proof. Recall that the linear system $|2\Theta|$ induces a map $f\colon \mathcal{J} \to \mathbb{P}^3$ whose image Kum_s is just the Kummer surface of $\mathcal{J}$, namely the quotient of $\mathcal{J}$ by the involution -1: so it is a nodal surface singular in $f(\mathcal{J}[2])$.

Given p not belonging to $\mathcal{J}[2]$ there exists a unique curve $C \in |2\Theta|$ singular in p: indeed since f is étale outside $\mathcal{J}[2]$, C is singular in p if and only if $f(C)$ is singular in $f(p)$ and there is a unique plane section of Kum_s singular in $f(p)$, namely the one obtained intersecting Kum_s with the plane tangent to Kum_s in p. Since it is easily proved that there exists a curve of the form $C = \Theta_x + \Theta_{-x}$ (for all x such a curve belong to $|2\Theta|$) singular in p, it is the unique one. If $C \cap \mathcal{J}[2] = \emptyset$ and C singular in p, then it is singular also in $-p$: therefore $\Theta_x \cap \Theta_{-x}$ consists of, at least, 2 points. Moreover, since $\Theta^2 = 2$, if $\Theta_x \cap \Theta_{-x}$ contained a third point we could conclude $\Theta_x = \Theta_{-x}$ thus $x \in \mathcal{J}[2]$ and $x \in C$. This proves item 1).

If $p \in \mathcal{J}[2]$ belongs to C then $f(p)$ is a node of Kum_s and $f(C) = H \cap Kum_s$, H being a plane of $\mathbb{P}^3$ passing through $f(p)$. The normal cone to Kum_s in $f(p)$ is the cone over a conic: if H intersects this cone in 2 distinct lines, $f(C)$ has a node in $f(p)$ and C, being a $[2:1]$ covering ramified in $f(p)$, has a node in p. The planes containing $f(p)$ and not intersecting the normal cone in 2 distinct lines are parametrized by a conic: thus the locus of $|2\Theta|$ parametrizing curves passing through p and not having a node in p is contained in a conic. On the other hand the curves of the form $\Theta_{p+x} + \Theta_{p-x}$ with $x \in \Theta$ never have a node in p: in fact, for $x \in \Theta$, an easy computation on $Pic(C^0)$ shows that either $\Theta_{p+x} \cap \Theta_{p-x} = p$ or $x = -x$ and $\Theta_{p+x} = \Theta_{p-x}$. Since, as already showed, any $C \in |2\Theta|$ singular outside $\mathcal{J}[2]$ is of the form $\Theta_x + \Theta_{-x}$, item (2) follows.

□

Since $NS(\mathcal{J}) = \mathbb{Z}\Theta$ the geometric genus of curve of $\mathcal{J}$ is at least 2. Since a curve C in $|2\Theta|$ has arithmetic genus 5, it can contain at most 3 nodes. Using this remark the previous Lemma can be reformulated as follows.

Proposition 2.1.3. *If $NS(\mathcal{J}) = \mathbb{Z}\Theta$, the stratification of $|2\Theta|$ by the analytic type of singularity is the following:*

- *Stratum S: the locus parametrizing smooth curves of genus 5.*
- *Stratum N(1): the locus parametrizing nodal irreducible curves singular in a unique 2-torsion point.*
- *Stratum N(2): the locus parametrizing nodal irreducible curves singular only in 2 distinct 2-torsion points.*
- *Stratum N(3): the locus parametrizing nodal irreducible curves singular only in 3 distinct points of 2-torsion.*
- *Stratum R(1): the locus parametrizing reducible curves with nodal singularity (they are the curves of the form $\Theta_x \cup \Theta_{-x}$ with 2 singular points, namely $\Theta_x \cap \Theta_{-x}$ consists of 2 distinct points).*
- *Stratum R(2): the locus parametrizing reducible curves with a unique singular point (they are the curves of the form $\Theta_x \cup \Theta_{-x}$ with a unique (non nodal) singular point, namely $\Theta_x \cap \Theta_{-x}$ consists of a unique point belonging to $\mathcal{J}[2]$).*
- *Stratum D: the locus parametrizing non reduced curves (they are the curves of the form $2\Theta_x$ with x a point of 2-torsion).*

In order to determine the Euler characteristic of $\widetilde{\mathcal{M}}$ we need to compute the Euler characteristic of the fibers of Φ and establish their dimension.

Proposition 2.1.4. *For any $C \in |2\Theta|$ the dimension of $\Phi^{-1}(C)$ is 3.*
If $C \in S \cup N(1) \cup N(2) \cup R(1) \cup R(2)$ then $\chi(\Phi^{-1}(C)) = 0$.
If $C \in N(3)$ then $\chi(\Phi^{-1}(C)) = 4$.
If $C \in D$ then $\chi(\Phi^{-1}(C)) = 20$.

Proof. In this proof we will denote by M_C, the locus of $\mathbf{M}_{(0,2\Theta,-2)}$ parametrizing sheaves whose Fitting subscheme is C: in particular $\Phi^{-1}(C) = M_C \cap \mathbf{M}^0_{(0,2\Theta,-2)}$.

If $C \in S \cup N(1) \cup N(2) \cup N(3)$, then points of M_C correspond to isomorphism classes of rank-1, torsion-free sheaves on C whose Euler characteristic is -2. It is known (see [Be 99]) that such a sheaf is either a degree-2 line bundle on C or the push-forward of a degree-(2-r) line bundle from a partial normalization desingularizing exactly r nodes of C.

Since line bundles having fixed degree on a nodal curve with n nodes are parametrized by a $(\mathbb{C}^*)^r - bundle$ on the Jacobian of its normalization (see [HM 98]), M_C can be stratified in $(\mathbb{C}^*)^r - bundle$ over the Jacobian $J(\widetilde{C})$ of the normalization $\widetilde{C}$.

Letting C' be a partial normalization of C we now determine the intersection of the stratum $U(C')$ parametrizing push-forward of line bundle on C' with $\Phi^{-1}(C)$. The restriction $r:$ $U(C') \to \mathcal{J}$ of the map $a_{(0,2\Theta,-2)}$ to $U(C')$ associates to a sheaf F, the point $\sum n_i p_i + \sum q_k$ where $q_k \in \mathcal{J}[2] \cap C$ are points having 2 distinct inverse images on C' and $\sum n_i p_i$ is the push-forward on $\mathcal{J}$ of a representative, in the Chow ring of $\widetilde{C}$, of c_1 of the pull-back of F.

It follows that r descends to a map from $J(\widetilde{C})$: moreover this last map is easily checked to be identified with the map

$$(7) \qquad\qquad a \colon J(\widetilde{C}) \to \mathcal{J}$$

induced on the Albanese varieties by the morphism $\widetilde{C} \to J$ obtained composing the normalization morphism of C with the inclusion of C in $\mathcal{J}$. $U(C')$ is then a $(\mathbb{C}^*)^r - bundle$ over the

fiber of a: if $C \in S \cup N(1) \cup N(2)$ then $dim(J(\widetilde{C})) > dim(\mathcal{J})$ and the Euler characteristic of a fiber of a is 0: it follows that $\chi(U(C')) = 0$ too.

Since $\Phi^{-1}(C)$ is stratified by subvarieties of the form $U(C')$, by the additivity of the Euler characteristic, we get $\chi(\Phi^{-1}(C)) = 0$.

If $C \in N(3)$ the strata of $\Phi^{-1}(C)$ parametrizing sheaves which are not push-forward of line bundle from $\widetilde{C}$, are bundles with fiber $(\mathbb{C}^*)^r$ with $r > 0$: their Euler characteristics are still 0. Using again the additivity of the Euler characteristic we get that $\chi(\Phi^{-1}(C))$ equals the Euler characteristic of the stratum parametrizing sheaves which are push-forward of line bundle from $\widetilde{C}$.

This stratum is, as already explained, in bijective correspondence with $a^{-1}(0)$. Since $C \in N(3)$, $J(\widetilde{C})$ is an abelian surface and since $NS(\mathcal{J}) = \mathbb{Z}\Theta$, $\mathcal{J}$ doesn't contain elliptic curves: therefore $a \colon J(\widetilde{C}) \to \mathcal{J}$ is an étale covering and $J(\widetilde{C})$ has the same Hodge structure over $\mathbb{Q}$ of $\mathcal{J}$, in particular letting $\Theta_{\widetilde{C}}$ be the theta divisor on $J(\widetilde{C})$ we get $NS(J(\widetilde{C})) = \mathbb{Z}\Theta_{\widetilde{C}}$. The degree of a is easily computed to be 4: in fact $a^*(\Theta) \cap \Theta_{\widetilde{C}} = C \cap \Theta = 4 = 2\Theta_{\widetilde{C}}^2$, hence $a^*(\Theta) = 2\Theta_{\widetilde{C}}$ and for $C \in N(3)$

$$(8) \qquad\qquad \chi(\Phi^{-1}(C)) = deg(a) = \frac{(2\Theta_{\widetilde{C}})^2}{(\Theta)^2} = 4.$$

Moreover, using again that $\mathcal{J}$ doesn't contain elliptic curves we get that a is always surjective, hence the fiber of a has dimension equal to the geometric genus of C minus 2: it follows from the given description of the stratification of $\Phi^{-1}(C)$, that for any $C \in S \cup N(1) \cup N(2) \cup N(3)$ the stratum of $\Phi^{-1}(C)$ parametrizing line bundles on C has dimension 3 and its complement has lower dimension.

We now consider the case $C \in R(1) \cup R(2)$. In this case $C = \Theta_x + \Theta_{-x}$. We will denote by $i_x \colon C^0 \to \mathcal{J}$ and $i_{-x} \colon C^0 \to \mathcal{J}$ the embeddings of C^0 on Θ_x and Θ_{-x} respectively.

Since the Fitting subscheme of a sheaf contains the subscheme defined by the annihilator of the sheaf (see [Ei 95]), for any sheaf F such that $[F] \in \Phi^{-1}(C)$ is the push-forward via the inclusion $i \colon C \to \mathcal{J}$ of a sheaf F_C on C: moreover since $c_1(F) = 2\Theta$, the restriction of F_C to each of the component of C is a rank-1 sheaf.

By the description of strictly semistable sheaves given in the proof of 1.1.8, points the locus of M_C parametrizing strictly semistable sheaves are in bijective correspondence with isomorphism classes of sheaves of the form $G_1 \oplus G_2$, where G_i are stable sheaves whose Mukai vector is $(0, c_1(\Theta), -1)$, namely $G_1 = (i_x)_* L_1$ and $G_2 = (i_{-x})_* L_2$ where L_i are degree-0 line bundles on C^0. With these notations, fixed $L_1 \in Pic^0(C^0)$, since the map associating to each line bundle L on C^0 the point $\sum n_i p_i \in \mathcal{J}$ (where $\sum n_i p_i$ is a representative of $c_2((i_x)_* L)$ in the Chow ring) is obviously an isomorphism, we get that there exists a unique L_2 such that $[(i_x)_* L_1 \oplus (i_{-x})_* L_2] \in \Phi^{-1}(C)$. It follows that the strictly semistable locus $\Phi^{-1}(C)^{ss} \subset \Phi^{-1}(C)$ is isomorphic to $\mathcal{J}$.

To study the stable locus $\Phi^{-1}(C)^s$, for any F having C as Fitting subscheme, we denote by L_1 and L_2 the torsion free parts of $i_x^*(F)$ and $i_{-x}^*(F)$. There are natural surjective maps from F to $(i_x)_* L_1$ and to $(i_{-x})_* L_2$: their direct sum is the first map in the following exact sequence

$$(9) \qquad\qquad 0 \to F \xrightarrow{\alpha} (i_x)_* L_1 \oplus (i_{-x})_* L_2 \xrightarrow{\beta} Q \to 0.$$

Since both the components of α are surjective Q is a quotient of both $(i_x)_* L_1$ and $(i_x)_* L_2$ and therefore a quotient of $\mathcal{O}_{\Theta_x \cap \Theta_{-x}}$. If F is stable then $deg(L_i) \geq 1$, and using (9) to compute Chern classes of F we find $deg(L_1) + deg(L_2) = lenght(Q)$: is then easy to check that $lenght(Q) = 2$ ($Q = \mathcal{O}_{\Theta_x \cap \Theta_{-x}}$) and $deg(L_2) = deg(L_1) = 1$. It follows that if F is stable it is the kernel of a map $\beta \colon (i_x)_* L_1 \oplus (i_x)_* L_2 \to \mathcal{O}_{\Theta_x \cap \Theta_{-x}}$, where L_i is a degree-1 line bundle and

the restriction of β to each summand is already surjective.

On the other hand any such a kernel is easily seen to be a stable sheaf, and, given 2 kernels obtained in this way, they are isomorphic if and only if they differ by an automorphism of $(i_{-x})_*L_1 \oplus (i_x)_*L_2$.

Fixed L_1 and L_2 it can be seen the following: if $C \in R(1)$ the kernels obtained are parametrized by $\mathbb{C}^* \times \mathbb{C}^*$ and the isomorphism classes of sheaves simply by $\mathbb{C}^*$, if $C \in R(2)$ the kernels obtained are parametrized by $\mathbb{C}^* \times \mathbb{C}$ and the isomorphism classes of sheaves simply by $\mathbb{C}$.

As in the strictly semistable case we can see that, fixed L_1 there exists a unique L_2 giving kernels belonging to $\Phi^{-1}(C)$.

It follows that, for $C \in R(1) \cup R(2)$, $\Phi^{-1}(C)^s$ is either a $\mathbb{C}^*$-bundle or a $\mathbb{C}$-bundle over $\mathcal{J}$. Since $\Phi^{-1}(C)$ is the disjoint union of $\Phi^{-1}(C)^s$ and $\mathcal{J}$, we get $\chi(\Phi^{-1}(C)) = 0$ and $dim(\Phi^{-1}(C)) = 3$.

It remains to consider the case $C \in D$. To simplify the notation we deal explicitly only with the case $C = 2\Theta$, but the same proof works with C replaced by any double curve in $|2\Theta|$.

In this case the subscheme defined by the annihilator of a sheaf F such that $[F] \in \Phi^{-1}(C)$, being a pure 1-dimensional scheme contained in C is either Θ or the double curve C.

If the annihilator of F (such that $[F] \in \Phi^{-1}(C)$) is the ideal I_Θ of Θ then $F = (i_0)_*V$, where V is a semistable rank-2 vector bundle with trivial determinant and $i_0 \colon C^0 \to \mathcal{J}$ is the imbedding of C^0 on Θ. The locus of $\Phi^{-1}(C)$ parametrizing sheaves of this form is then isomorphic to the moduli space of rank-2 semistable vector bundles with fixed determinant on a genus 2 curve: Narashiman and Ramanan proved in [NR 69] that it isomorphic to $\mathbb{P}^3$.

Since any polystable sheaf F such that $[F] \in \Phi^{-1}(C)$ is the push-forward of a rank-2 vector bundle from C, points of the complement U of this $\mathbb{P}^3$ are in [1:1] correspondence with isomorphism classes of stable sheaves annihilated by the ideal I_C of C.

Letting F be such a sheaf we want to prove that it fits in an exact sequence

$$(10) \qquad 0 \to (i_0)_*(K^\vee \otimes L) \to F \to (i_0)_*L \to 0.$$

where K is the canonical bundle and L is a degree-1 line bundle such that $L^{\otimes 2} \otimes K^\vee = \mathcal{O}_{C^0}$. Making the Tensor product of F with the exact sequence of sheaves on $\mathcal{J}$ defining the double structure of C

$$(11) \qquad 0 \to (i_0)_*K^\vee \to \mathcal{O}_C \to \mathcal{O}_{2\Theta} \to 0.$$

we get the exact sequence

$$(12) \qquad (i_0)_*(K^\vee \otimes L \oplus T) \to F \to (i_0)_*(L \oplus T) \to 0.$$

where L is a vector bundle and T a torsion sheaf on C^0. Since $c_1(F) = 2\Theta$ and F is not the push-forward of a sheaf from C^0, the rank of L is 1. Since F is stable and F surjects on $(i_0)_*L$, we have $deg(L) > 0$. On the other hand since F is pure 1-dimensional the kernel of the first map in the last exact sequence is just T. Since $ch_2(F) = -2$ we get $2deg(L) + length(T) = 2$: this implies $deg(L) = 1$ and $T = 0$, so proving the existence of the exact sequence (10) ($L^{\otimes 2} \otimes K^\vee = \mathcal{O}_{C^0}$ is required to have $[F] \in \Phi^{-1}(C)$). Moreover it is easily seen that for any sheaf F, having the ideal I_C as its annihilator and fitting in such an exact sequence a pure 1-dimensional sheaf and the subsheaf $(i_0)_*(K^\vee \otimes L)$ is just the subsheaf annihilated by the ideal I_Θ of Θ: since stabilty can be checked using only injections of push-forwards of line bundle on C^0 and these are always annihilated by I_Θ it follows that any F under our condition is stable. We can conclude that for any L satisfying $L^{\otimes 2} \otimes K^\vee = \mathcal{O}_{C^0}$ there is a locus $U_L \subset U$ whose points are in bijective correspondence with isomorphism classes of extensions of the form (10) with F not annihilated by I_Θ. Using Hirrzebruch-Riemann-Roch and Serre duality for extensions (see [HL 97]) we get $dim(Ext^1((i_0)_*L, (i_0)_*(K^\vee \otimes L))) = 4$ and the extensions being push-forward of vector bundles can be identified using the spectral sequence associated to the composition

$H^0 \circ \mathcal{H}om(i_*L, \cdot) = Hom((i_0)_*L,)$: it produces the following short exact sequence

$$0 \to H^1(\mathcal{H}om((i_0)_*L, (i_0)_*(K^\vee \otimes L))) \to$$

$$Ext^1((i_0)_*L, (i_0)_*(K^\vee \otimes L)) \to H^0(\mathcal{E}xt^1((i_0)_*L, (i_0)_*(K^\vee \otimes L))) \to 0.$$

where the first term is isomorphic to $\mathbb{C}^3$ and parametrizes just the extensions coming from extensions of line bundles on C^0.

Since for any extension belonging to $Ext^1((i_0)_*L, (i_0)_*(K^\vee \otimes L)) \setminus H^1(\mathcal{H}om((i_0)_*L, (i_0)_*(K^\vee \otimes L)))$ we have $End((i_0)_*(K^\vee \otimes L)) = End((i_0)_*L) = End(F) = \mathbb{C}$, 2 such extensions have middle terms isomorphic if and only if they differ by a scalar multiplication: it follows that U_L is in bijective correspondence with $\mathbb{C}^3$.

Since there are 16 line line bundles on C^0 satisfying $L^{\otimes 2} \otimes K^\vee = \mathcal{O}_{C^0}$ and the respective $U'_L s$ are easily seen to be disjoint we get that $\Phi^{-1}(C)$ is the disjoint union of $\mathbb{P}^3$ with 16 3-dimensional affine spaces: therefore $dim(\Phi^{-1}(C)) = 3$ and $\chi(\Phi^{-1}(C)) = 20$. $\square$

Before calculating the Euler characteristic of $\mathbf{M}^0_{(0,2\Theta,-2)}$ we establish its irreducibility as a corollary of the previous proposition.

Corollary 2.1.5. $\mathbf{M}^0_{(0,2\Theta,-2)}$ and $\mathbf{M}_{(0,2\Theta,-2)}$ are reduced irreducible.

Proof. Since $a_{(0,2\Theta,-2)}$ makes $\mathbf{M}_{(0,2\Theta,-2)}$ a fibration over $\widehat{\mathcal{J}} \times \mathcal{J}$ with fiber $\mathbf{M}^0_{(0,2\Theta,-2)}$, the second statement follows from the first.

Let's prove that $\mathbf{M}_{(0,2\Theta,-2)}$ is reduced irreducible.

Since by Proposition 1.1.8 $\mathbf{M}^0_{(0,2\Theta,-2)}$ is reduced purely 6-dimensional, it is enough to prove that there exists an irreducible open subvariety $U \subset \mathbf{M}^0_{(0,2\Theta,-2)}$ whose complement has dimension at most 5.

We set $U := \Phi^{-1}(S)$. By the previous proposition we have $dim(\mathbf{M}^0_{(0,2\Theta,-2)} \setminus U) \geq 5$. To show the irreducibility of U recall that for $C \in S$ the fiber $\Phi^{-1}(C)$ is identified to the kernel of the map

$$a : J(C) := \frac{H^1(\Omega_C)^\vee}{H_1(C, \mathbb{Z})} \to \frac{H^1(\Omega_{\mathcal{J}})^\vee}{H_1(\mathcal{J}, \mathbb{Z})}$$

induced on the Albanese varieties by the embedding of C in $\mathcal{J}$. The Kernel of a is an irreducible torus because C is an ample divisor and hence, by the Hyperplane section theorem, $H_1(C, \mathbb{Z})$ surjects on $H_1(\mathcal{J}, \mathbb{Z})$. U is finally irreducible being a bundle with irreducible fiber and base. $\square$

We can now compute the Euler characteristic of $\mathbf{M}^0_{(0,2\Theta,-2)}$.

Proposition 2.1.6. $\chi(\mathbf{M}^0_{(0,2\Theta,-2)}) = 1280$.

Proof. By Proposition 2.1.3 and the additivity of Euler characteristic we get $\chi(\mathbf{M}^0_{(0,2\Theta,-2)}) = \chi(\Phi^{-1}(S \cup N(1) \cup N(2) \cup R(1) \cup R(2)) + \chi(D \cup N(3))$ and since the Euler characteristic of any fiber in $\Phi^{-1}(S \cup N(1) \cup N(2) \cup R(1) \cup R(2)$ is 0, we obtain $\chi(\Phi^{-1}(S \cup N(1) \cup N(2) \cup R(1) \cup R(2)) = 0$ (see [Be 99]). Letting $|N(3)|$ be the cardinality of $N(3)$ and using again Proposition 2.1.4 we conclude

$$(13) \qquad \chi(\mathbf{M}^0_{(0,2\Theta,-2)}) = \chi(\Phi^{-1}(D)) + \chi(\Phi^{-1}(n(3))) = 16 \cdot 20 + 4|N(3)|.$$

$|N(3)|$ is computed as follows. A triple of singular points of Kum_s defines a curve in $N(3)$ if and only if it is not included in a double curve: thus

$$(14) \qquad |N(3)| = \binom{16}{3} - 16\binom{6}{3} = 240,$$

$\binom{6}{3}$ are the triples of singular points included in a double curve.
Formulas (13) and (14) and Proposition 2.1.4 imply the result. $\qquad\square$

Since we know the birational modification needed to obtain $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ from $\mathbf{M}^0_{(0,2\Theta,-2)}$ it is now possible to compute $\chi(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)})$.

Theorem 2.1.7. $\chi(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}) = 1920$.

Proof. Since $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ is the disjoint union of the stable locus of $\mathbf{M}^0_{(0,2\Theta,-2)}$ and $\widetilde{\Sigma}^0_{(0,2\Theta,-2)}$ and by the additivity of χ we have

$$\chi(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}) = \chi(\mathbf{M}^0_{(0,2\Theta,-2)}) - \chi(\Sigma^0_{(0,2\Theta,-2)}) + \chi(\widetilde{\Sigma}^0_{(0,2\Theta,-2)}).$$

Since the map $\mathcal{J} \times \widehat{\mathcal{J}} \to \Sigma^0_{(0,2\Theta,-2)}$, associating to (x,L) the the s-equivalence class of $i_{x*}L \oplus i_{-x*}L$, is surjective and [2:1] outside the 256 2-torsion points of $\mathcal{J} \times \widehat{\mathcal{J}}$ we find

$$\chi(\Sigma^0_{(0,2\Theta,-2)}) = \frac{\chi(\mathcal{J} \times \widehat{\mathcal{J}}) + 256}{2} = 128.$$

To compute $\chi(\widetilde{\Sigma}^0_{(0,2\Theta,-2)})$ recall (see remark 1.1.10) that the restriction to $\widetilde{\Sigma}^0_{(0,2\Theta,-2)}$ of the desingularization map is a $\mathbb{P}^1$ bundle outside the 256 points of $\Omega^0_{(0,2\Theta,-2)}$ where the fibers are smooth 3-dimensional quadrics. The Euler characteristic of the $\mathbb{P}^1$ bundle is -256 and the Euler characteristic of the 3-dimensional quadric is 4: the final result is

$$\chi(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}) = 1280 - 128 - 256 + 256 \cdot 4 = 1920.$$

$\qquad\square$

2.2. $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ is a birational model of $\widetilde{\mathcal{M}}$. In this section we show that $\chi(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}) = \chi(\widetilde{\mathcal{M}})$.

This equation is a consequence of the following proposition.

Proposition 2.2.1. *There exists a birational map $b : \widetilde{\mathcal{M}} \dashrightarrow \widetilde{\mathbf{M}}^0_{(0,2\Theta,2)}$.*
Moreover, letting b^* be the pull-back of divisors, $b^*\widetilde{\Sigma}^0_{(0,2\Theta,-2)} = \widetilde{\Sigma}$.

Proof. Since $\mathbf{NS}(\mathcal{J}) = \mathbb{Z}\Theta$ the tensorization by $\mathcal{O}(\Theta)$ doesn't change stability and semistability, hence it induces an isomorphism

$$t : \mathbf{M}_{(2,0,-2)} \to \mathbf{M}_{(2,2\Theta,0)}.$$

An easy calculation shows that this isomorphism sends $\Sigma_{(2,0,-2)}$ to $\Sigma_{(2,2\Theta,0)}$ and $\mathbf{M}^0_{(2,0,-2)}$ to $\mathbf{M}^0_{(2,2\Theta,0)}$.
Recalling that, for v as in Proposition 1.1.11 the blow up $Bl_{\Sigma^0_v \setminus \Omega^0_v}\mathbf{M}^0_v \setminus \Omega^0_v$ is an open subset of $\widetilde{\mathbf{M}}^0_v$ and its complement has codimension bigger than 1 (see remark 1.1.6), we get that t induces a birational map (actually biregular) $\widetilde{t} : \widetilde{\mathcal{M}} \to \widetilde{\mathbf{M}}^0_{(2,2\Theta,0)}$ such that $\widetilde{t}^*\widetilde{\Sigma}^0_{(2,2\Theta,0)} = \widetilde{\Sigma}$.
It remains to prove that there exists a birational map $\widetilde{fm} : \widetilde{\mathbf{M}}^0_{(2,2\Theta,0)} \dashrightarrow \widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ such that

$$\widetilde{fm}^*\widetilde{\Sigma}^0_{(0,2\Theta,-2)} = \widetilde{\Sigma}^0_{(2,2\Theta,0)}.$$

We have denoted this map by $\widetilde{fm}$ because it is induced by the Fourier-Mukai transform (see remark 1.2.2)

$$\mathcal{FM} : D\mathcal{C}oh(\mathcal{J}) \to D\mathcal{C}oh(\mathcal{J}).$$

We need the following lemma.

Lemma 2.2.2. *Let F be a strictly semistable sheaf, having Mukai's vector $(2, 2\Theta, 0)$ (or $(0, 2\Theta, -2)$), then F verifies the W.I.T. (with index 1), and moreover $\mathcal{FM}(F)(-1)$ is a strictly semistable sheaf having Mukai's vector $(0, 2\Theta, -2)$ (or $(2, 2\Theta, 0)$).*

Proof. We only deal with the first case, the second being completely analogous. By the description of strictly semistable sheaves (sequence (3.3) in the proof of Proposition 1.1.8) it is enough to verify that a sheaf of the form $I_x \otimes \mathcal{O}(\Theta_y)$ verifies the W.I.T. (with index 1) and its Fourier-Mukai transform is a sheaf of the form $(i_z)_* L$ where $i_z : C^0 \to \mathcal{J}$ is the embedding on Θ_z and L is a degree-0 line bundle.

Indeed, since both $\mathbb{C}_x$ and $\mathcal{O}(\Theta_y)$ satisfy the weak index theorem with index 0 and their Fourier-Mukai transform are $\mathcal{O}(\Theta_x - \Theta)$ and $\mathcal{O}(\Theta_y)^\vee$ respectively (see Theorem 3.13 [Mu 81]), the short exact sequence

$$0 \to I_x \otimes \mathcal{O}(\Theta_y) \to \mathcal{O}(\Theta_y) \to \mathbb{C}_x \to 0 \; (*)$$

induces the long exact sequence

$$0 \to q_{2*}(\mathcal{P} \otimes q_1^*(I_x \otimes \mathcal{O}(\Theta_y))) \to \mathcal{O}(\Theta_y)^\vee \to \mathcal{O}(\Theta_x - \Theta) \to R^1 q_{2*}(\mathcal{P} \otimes q_1^*(I_x \otimes \mathcal{O}(\Theta_y))) \to 0$$

and since the middle map of this sequence cannot be zero, the first term is 0 and the last one is just $\mathcal{FM}(I_x \otimes \mathcal{O}(\Theta_y))(-1)$ and has the stated form.

$\square$

By general results on Fourier-Mukai transform (see [Mu 81] and [Mu 87]), and since $\mathbf{M}_{(2,2\Theta,0)}$ and $\mathbf{M}_{(0,2\Theta,-2)}$ are both reduced irreducible (see 2.1.5) this lemma implies that there exists a birational map $fm : \mathbf{M}_{(2,2\Theta,0)} \to \mathbf{M}_{(0,2\Theta,-2)}$ restricting to an isomorphism on a neighborhood of $\Sigma_{(0,2\Theta,-2)}$ and such that $fm(\Sigma_{(2,2\Theta,0)}) = \Sigma_{(0,2\Theta,-2)})$.

Recalling how O'Grady's desingularizations are obtained, we get a lift

$$\overline{fm} : \widetilde{\mathbf{M}}_{(2,2\Theta,0)} \to \widetilde{\mathbf{M}}_{(0,2\Theta,-2)}$$

such that $\overline{fm}^* \Sigma_{(0,2\Theta,-2)} = \Sigma_{(2,2\Theta,0)}$.

To complete the proof of this proposition it remains to show that $\overline{fm}$ sends fibers of $a_{(2,2\Theta,0)} : \widetilde{\mathbf{M}}_{(2,2\Theta,0)} \to \mathcal{J} \times \widehat{\mathcal{J}}$ birationally to fibers of $a_{(0,2\Theta,-2)} : \widetilde{\mathbf{M}}_{(0,2\Theta,-2)} \to \mathcal{J} \times \widehat{\mathcal{J}}$.

Since the fibers of $a_{(2,2\Theta,0)}$ are isomorphic to $\widetilde{\mathcal{M}}$, their fundamental group is trivial: hence $a_{(2,2\Theta,0)}$ is identified with the Albanese map of $\widetilde{\mathbf{M}}_{(2,2\Theta,0)}$. On the other hand the rational map $a_{(2,2\Theta,0)} \circ \overline{fm} : \widetilde{\mathbf{M}}_{(2,2\Theta,0)} \to \mathcal{J} \times \widehat{\mathcal{J}}$, being a map to an abelian variety, extends. By the universal property of the Albanese map there exists a morphism g making commutative the following diagram:

$$
\begin{array}{ccc}
\widetilde{\mathbf{M}}_{(2,2\Theta,0)} & \xrightarrow{\overline{fm}} & \widetilde{\mathbf{M}}_{(0,2\Theta,-2)} \\
a_{(2,2\Theta,0)} \downarrow & & a_{(0,2\Theta,-2)} \downarrow \\
\mathcal{J} \times \widehat{\mathcal{J}} & \xrightarrow{g} & \mathcal{J} \times \widehat{\mathcal{J}}
\end{array}
$$

Thus $\overline{fm}$ sends fibers of $a_{(2,2\Theta,0)}$ to fibers of $a_{(0,2\Theta,-2)}$. On the other hand we already proved (Corollary 2.1.5) that fibers of $a_{(0,2\Theta,-2)}$ are irreducible. It follows that $\overline{fm}$ induces, by restriction to the central fibers, a map $\widetilde{fm} : \widetilde{\mathbf{M}}^0_{(2,2\Theta,0)} \to \widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ such that $\widetilde{fm}^*(\Sigma^0_{(0,2\Theta,-2)}) = \Sigma^0_{(2,2\Theta,0)}$. The map $b := \widetilde{fm} \circ \widetilde{t}$ verifies the thesis of the proposition. $\square$

Theorem 2.2.3. $\chi(\widetilde{\mathbf{M}}^0_{(2,0,-2)}) = 1920$

Proof. By Proposition 1.1.11 $\widetilde{\mathcal{M}}$ is symplectic and projective. By the previous proposition it is birational to an irreducible symplectic variety: it follows that $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ is an irreducible symplectic variety too.

By a theorem due to Huybrechts ([Hu 97]) birational irreducible symplectic varieties are deformation equivalent: therefore $\chi(\widetilde{\mathcal{M}}) = \chi(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)})$ and the last is 1920 as shown in Theorem 2.1.7. $\qquad\square$

3. The Beauville form of $\widetilde{\mathcal{M}}$

In this section we determine the Beauville form and the Fujiki constant of $\widetilde{\mathcal{M}}$ (see Theorem 3.5.1). Before going on we recall the Theorem due to Beauville and Fujiki (see [Be 83, Fu 87]) that defines the the Beauville form and the Fujiki constant of an irreducible symplectic variety.

Theorem 3.0.4. *Let X be a 2n dimensional irreducible symplectic manifold. There exist a unique indivisible bilinear integral symmetric form $B_X \in S^2(H^2(X,\mathbb{Z}))^*$ and a unique positive constant $c_X \in \mathbb{Q}$ such that for any $\alpha \in H^2(X,\mathbb{Z})$*

$$(15) \qquad \int_X \alpha^{2n} = c_X B_X(\alpha,\alpha)^n$$

and for $0 \neq \omega \in H^0(\Omega^2_X)$

$$(16) \qquad B_X(\omega + \overline{\omega}, \omega + \overline{\omega}) > 0.$$

Definition 3.0.5. The quadratic form B_X of the previous theorem is the Beuville form. The constant c_X is the Fujiki constant.

Remark 3.0.6. The formula (15) is named the Fujiki formula; its polarized form is the following

$$(17) \qquad \int_X \alpha_1 \wedge \dots \wedge \alpha_{2n} = \frac{c_X}{2n!} \sum_{\sigma \in S_{2n}} B(\alpha_{\sigma(1)}, \alpha_{\sigma(2)})\dots B(\alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)})$$

This section is organized as follows. In subsection 1 we simply recall the basis of $H^2(\widetilde{\mathcal{M}}, \mathbb{Q})$ given by O'Grady. Subsections 2, 3 and 4 are devoted to extract from O'Grady's basis a basis of $H^2(\widetilde{\mathcal{M}}, \mathbb{Z})$. In subsection 2 it is shown that the submodule $\widetilde{\mu}(H^2(\mathcal{J}, \mathbb{Z}))$ obtained by means of the Donaldson morphism (see Definition 3.1.1) is saturated (Proposition 3.2.1). In subsection 3 it is proved that 2 divides $c_1(\widetilde{\Sigma})$ in $H^2(\widetilde{\mathcal{M}}, \mathbb{Z})$ (Theorem 3.3.1). Using these results and letting $\widetilde{B}$ be the strict transform in $\widetilde{\mathcal{M}}$ of the locus parametrizing stable sheaves non locally free, we can easily prove Theorem 3.4.1 of subsection 4, asserting that

$$H^2(\widetilde{\mathcal{M}}, \mathbb{Z}) = \mathbb{Z}\frac{c_1(\widetilde{\Sigma})}{2} \oplus \mathbb{Z}c_1(\widetilde{B}) \oplus \widetilde{\mu}(H^2(\mathcal{J}, \mathbb{Z})).$$

Finally in subsection 5 few intersection numbers on $\widetilde{\mathcal{M}}$ are needed to completely determine the Beauville form.

3.1. The rational basis. In this subsection we simply recall the basis of $H^2(\widetilde{\mathcal{M}}, \mathbb{Q})$ given by O'Grady.

There exists a smaller compactification, namely the Uhlenbeck compactification, $\mathbf{M}^U$ of the $\mu-$stable locus of $\mathbf{M}^0_{(2,0,-2)}$. Moreover $\mathbf{M}^U$ is endowed with a surjective map

$$\varphi : \mathbf{M}^0_{(2,0,-2)} \longrightarrow \mathbf{M}^U.$$

Associated with φ there is a cohomological linear map, the Donaldson morphism

(18) $$\mu : H^2(\mathcal{J}, \mathbb{Z}) \longrightarrow H^2(\mathbf{M}^U, \mathbb{Z})$$

having the following property (see [Li 93], [FM 94] and [Mo 93]).
Given $\mathcal{F}$ a flat family of sheaves in $\mathbf{M}^0_{(2,0,-2)}$, parametrized by a scheme X, letting

$$f_{\mathcal{F}} : X \to \mathbf{M}^0_{(2,0,-2)}$$

be the modular morphism and letting p and q be the two projections to $\mathcal{J}$ and X respectively

(19) $$(f_{\mathcal{F}}^* \circ \varphi^* \circ \mu)(\alpha) = q_*(p^*(\alpha) \cup c_2(\mathcal{F}))$$

for any $\alpha \in H^2(\mathcal{J}, \mathbb{Z})$.
We can finally define

Definition 3.1.1.
$$\widetilde{\mu} := \widetilde{\pi}^* \circ \varphi^* \circ \mu : H^2(\mathcal{J}, \mathbb{Z}) \to H^2(\widetilde{\mathcal{M}}, \mathbb{Z}).$$

In order to recall the basis of $H^2(\widetilde{\mathcal{M}}, \mathbb{Q})$ we need to recall the definition of a divisor on $\widetilde{\mathcal{M}}$.

Definition 3.1.2. Let $B \subset \mathbf{M}^0_{(2,0,-2)}$ be the locally closed subset parametrizing non locally free stable sheaves, O'Grady defines

$$\widetilde{B} := \overline{\widetilde{\pi}^{-1}(B)}.$$

We are now ready to present the rational basis: the following is Proposition (7.3.3) of [OG 03].

Proposition 3.1.3. *The homomorphism* $\widetilde{\mu} : H^2(\mathcal{J}, \mathbb{Z}) \to H^2(\widetilde{\mathcal{M}}, \mathbb{Z})$ *is injective and*

$$\widetilde{\mu}(H^2(\mathcal{J}, \mathbb{Q})), \quad \mathbb{Q}c_1(\widetilde{\Sigma}), \quad \mathbb{Q}c_1(\widetilde{B})$$

are linearly independent subspaces of $H^2(\widetilde{\mathcal{M}}, \mathbb{Q})$. *Moreover*

$$\widetilde{\mu}(H^2(\mathcal{J}, \mathbb{Q})) \oplus \mathbb{Q}c_1(\widetilde{\Sigma}) \oplus \mathbb{Q}c_1(\widetilde{B}) = H^2(\widetilde{\mathcal{M}}, \mathbb{Q})$$

We will start from the given basis to find 8 independent generators of $H^2(\widetilde{\mathcal{M}}, \mathbb{Z})$.

3.2. **The image of the Donaldson's morphism.** Our first result in the study of the 2-cohomology of $\widetilde{\mathcal{M}}$ is the following.

Proposition 3.2.1. *The image of* $\widetilde{\mu} := \widetilde{\pi}^* \circ \varphi^* \circ \mu : H^2(\mathcal{J}, \mathbb{Z}) \longrightarrow H^2(\widetilde{\mathcal{M}}, \mathbb{Z})$ *is a saturated submodule.*

Proof. By simple linear algebra it is enough to prove that there exists

$$\alpha : H^2(\widetilde{\mathcal{M}}, \mathbb{Z}) \longrightarrow \mathbb{Z}^n$$

such that the restriction of α to $\widetilde{\mu}(H^2(\mathcal{J}, \mathbb{Z}))$ is injective and $\alpha(\widetilde{\mu}(H^2(\mathcal{J}, \mathbb{Z})))$ is a saturated submodule. Our map α will be the pull back map associated to a closed embedding $\widetilde{B}_p \subset \widetilde{\mathcal{M}}$ that we are going to define.
Given $p \in \mathcal{J} \setminus \mathcal{J}[2]$ let B_p be the locus of $\mathbf{M}^0_{(2,0,-2)}$ parametrizing non locally free stable sheaves, with singularities in p or $-p$: B_p is locally closed but not closed. We define:

$$\widetilde{B}_p := \overline{\widetilde{\pi}^{-1}B_p}$$

namely $\widetilde{B}_p$ is the strict transform of B_p in $\widetilde{\mathcal{M}}$. $\widetilde{B}_p$ has already been described in [OG 03]: the following proposition can be, almost completely, extracted from subsection 5.1 of that paper:

Proposition 3.2.2. (1) B_p *parametrizes semistable sheaves* N *fitting into exact sequences of the form*

$$(20) \qquad 0 \to N \to V \to \mathbb{C}_p \oplus \mathbb{C}_{-p} \longrightarrow 0$$

where V *belongs to the moduli space* Muk *of Mukai-stable (see* [BDL 01]*) rank two vector bundles with trivial determinant and* c_2^{hom}, *namely either* $V = L \oplus L^\vee$ *with* $L \in \widehat{\mathcal{J}} \setminus \widehat{\mathcal{J}}[2]$ *or* V *represents a non trivial element in* $Ext^1(L, L)$ *with* $L \in \widehat{\mathcal{J}}[2]$.

(2) *The rational map*

$$\zeta : \quad \begin{array}{ccc} Muk & \to & K(\widehat{\mathcal{J}}) \\ L \oplus L^\vee & \mapsto & (L, L^\vee) \end{array}$$

extends to an isomorphism (here, $K(\widehat{\mathcal{J}})$ *is the Kummer surface of* $\widehat{\mathcal{J}}$, *namely the locus of* $Hilb^2(\widehat{\mathcal{J}})$ *whose points correspond to schemes with associated cycles summing up to* $0 \in \widehat{\mathcal{J}}$ *).*

(3) *The rational map*

$$\phi : \widetilde{B}_p \to Muk$$

defined on $\widetilde{\pi}^{-1}(B_p)$ *associating to* $x \in \widetilde{\pi}^{-1}([F])$ *the class* $[F^{\vee\vee}]$ *in* Muk *extends to a regular morphism.*

(4) *The composition of the extensions*

$$\psi := \zeta \circ \phi : \widetilde{B}_p \to K(\widehat{\mathcal{J}})$$

endows $\widetilde{B}_p$ *with the structure of a* $\mathbb{P}^1$*-bundle.*

(5) *Denote* $\widetilde{i} : \widetilde{B}_p \to \widetilde{\mathcal{M}}$ *the closed embedding, then there exists a map* i_U *making commutative the following diagram*

$$\begin{array}{ccc} \widetilde{B}_p & \xrightarrow{\ \widetilde{i}\ } & \widetilde{\mathcal{M}} \\ \psi \downarrow & & \varphi \circ \widetilde{\pi} \downarrow \\ K(\widehat{\mathcal{J}}) & \xrightarrow{\ i_U\ } & \mathbf{M}^U. \end{array}$$

Proof. (1),(3) and (4) are proved in Lemma (4.3.3) and in section 5.1 of [OG 03].

(2) is contained in Theorem 5.6 of [BDL 01].

(5) follows from the classification of the fibers of φ (see [HL 97]) since the fibers of ψ are contracted by $\varphi \circ \widetilde{\pi}$. $\qquad \square$

The proof Proposition 3.2.1 will follow easily from the following claim.

Claim 3.2.3. Let $b : Bl_{\mathcal{J}[2]}\mathcal{J} \to \mathcal{J}$ be the blow up of the $2-torsion$ points of $\mathcal{J}$, let $q : Bl_{\mathcal{J}[2]}\mathcal{J} \to K(\mathcal{J})$ be the quotient by the involution, let $e : K(\widehat{\mathcal{J}}) \to K(\mathcal{J})$ be the identification induced by the one between $\mathcal{J}$ and $\widehat{\mathcal{J}}$, then

$$\widetilde{i}^* \circ \widetilde{\mu}(H^2(\mathcal{J}, \mathbb{Z})) = \psi^* \circ e^* \circ q_* \circ b^*(H^2(\mathcal{J}, \mathbb{Z}))$$

Proof. The first step in proving the claim is to produce a 'complete' family $\mathcal{N}$ of sheaves in $\widetilde{\pi}(\widetilde{B}_p)$ and to study the topology of its base.

We start constructing a universal family $\mathcal{V}$ parametrized by $K(\mathcal{J})$ for sheaves which are middle terms of the sequences (20): consider in fact the structural sheaf of the tautological subvariety of $Hilb^2(\mathcal{J}) \times \mathcal{J}$ and restrict it to $K(\mathcal{J}) \times \mathcal{J}$. This sheaf $\mathcal{G}$ can be seen as a family of sheaves of length 2 quotients of $\mathcal{O}_{\mathcal{J}}$, they fit in exact sequences of the form

$$0 \to \mathbb{C}_x \to G \to \mathbb{C}_{-x} \to 0$$

(x is a point of $\mathcal{J}$), moreover such a sequence splits if and only if $x \neq -x$. Recall then that for any $x \in \mathcal{J}$ the sheaf $\mathbb{C}_x$ satisfies the weak index theorem [Mu 81] , therefore any G, being the middle term of such an extension, satisfies W.I.T. By the theory of [Mu 87], applying the Fourier-Mukai transform to the family $\mathcal{G}$, we obtain a new family $\mathcal{V}$ of sheaves on $\mathcal{J}$. Since the Fourier-Mukai transform is a self-equivalence of the derived category of $\mathcal{J}$ and the Fourier-Mukai transform of a sheaf $\mathbb{C}_x$ is $\mathcal{O}(\Theta_x - \Theta)$, $\mathcal{V}$ is a family parametrizing bijectively sheaves in Muk.

We can now construct the family $\mathcal{N}$: consider the relative Quot-scheme parametrizing 0-dimensional length 2 quotients of the 'fibers' of $\mathcal{V}$, take the closed subscheme Q of the sheaves whose support is exactly $\{p, -p\}$ and, in this subscheme, the open $\mathcal{U}$ corresponding to quotients having semistable kernels. On $\mathcal{J} \times \mathcal{U}$ we have the following exact sequence of $\mathcal{U}$-flat sheaves:

$$(21) \qquad\qquad 0 \longrightarrow \mathcal{N} \longrightarrow (id \times pr)^*\mathcal{V} \longrightarrow \mathcal{T} \longrightarrow 0$$

where pr is the natural projection from $\mathcal{U}$ to $K(\mathcal{J})$ and $\mathcal{T}$ and $\mathcal{N}$ respectively the restrictions of the tautological quotient and the tautological kernel of the previous Quot-scheme. More explicitly Q can be obtained as follows. Let q_i be the projection of $\mathcal{J} \times K(\mathcal{J})$ to its i-th factor: then $q_{2*}\mathcal{H}om(\mathcal{V}, q_1^*\mathbb{C}_p)$ and $q_{2*}\mathcal{H}om(\mathcal{V}, q_1^*\mathbb{C}_{-p})$ are rank 2-vector bundles on $K(\mathcal{J})$ and there is an identification

$$Q = \mathbb{P}(q_{2*}\mathcal{H}om(\mathcal{V}, q_1^*\mathbb{C}_p)) \times_{K(\mathcal{J})} \mathbb{P}(q_{2*}\mathcal{H}om(\mathcal{V}, q_1^*\mathbb{C}_{-p})).$$

In particular the fiber on $V \in Muk \simeq K(\mathcal{J})$ is $\mathbb{P}(Hom(V, \mathbb{C}_p)) \times \mathbb{P}(Hom(V, \mathbb{C}_{-p})) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. To understand $\mathcal{U}$ we have to detect the unstable locus of each fiber. Let $([l_p], [l_{-p}]) \in \mathbb{P}(Hom(V, \mathbb{C}_p)) \times \mathbb{P}(Hom(V, \mathbb{C}_{-p}))$, then $([l_p], [l_{-p}])$ gives an unstable N if and only if there exists $L \subset V$ such that $c_1(L) = 0$ with $l_p(L_p) = l_{-p}(L_{-p}) = 0$. It can be easily checked that, if V is a direct sum there are only 2 points giving N unstable and if V is a non trivial extension there is a unique point giving N unstable: therefore $\mathcal{U}$ is isomorphic to the complement of a 2 dimensional variety in the 4-fold Q.

The second step of the proof of the claim consists in computing the pull-back of $\varphi^* \circ \mu(H^2(\mathcal{J}, \mathbb{Z}))$ via the modular map $f_\mathcal{N}$ associated to the family $\mathcal{N}$. The result is the following:

$$(22) \qquad\qquad f_\mathcal{N}^* \circ \varphi^* \circ \mu(H^2(\mathcal{J}, \mathbb{Z})) = pr^* \circ q_* \circ b^*(H^2(\mathcal{J}, \mathbb{Z})).$$

Let p_i be the projection from $\mathcal{J} \times \mathcal{U}$ to the i-th factor, and apply Whitney's formula to the exact sequence (21), since the support of $\mathcal{T}$ is $p_1^{-1}(p) \cup p_1^{-1}(-p)$, hence $c_1(\mathcal{T}) = 0$, we get:

$$c_2(\mathcal{N}) = c_2((id \times pr)^*\mathcal{V}) - c_2(\mathcal{T}).$$

So for any α belonging to $H^2(\mathcal{J}, \mathbb{Z})$, we have

$$p_{2*}(p_1^*\alpha \cup c_2(\mathcal{N})) = p_{2*}(p_1^*\alpha \cup (c_2((id \times pr)^*\mathcal{V}) - c_2(\mathcal{T}))) = p_{2*}(p_1^*\alpha \cup c_2((id \times pr)^*\mathcal{V}))$$

where the second equality is verified because $c_2(\mathcal{T})$ is the pull back of a 4-form from $\mathcal{J}$. Since, obviously, $p_i = q_i \circ (id \times pr)$ we can simplify the last term of the last equation:

$$p_{2*}(p_1^*\alpha \cup c_2((id \times pr)^*\mathcal{V})) = p_{2*}((id \times pr)^*(q_1^*\alpha \cup c_2(\mathcal{V}))) = pr^*q_{2*}(q_1^*\alpha \cup c_2(\mathcal{V})).$$

Call now $\mathcal{I}$ the incidence subvariety of $\mathcal{J} \times K(\mathcal{J})$: as we said earlier, $\mathcal{V}$ is the Fourier-Mukai transform of the family $\mathcal{O}_\mathcal{I}$; let then $FM^\bullet$ be the cohomological Fourier-Mukai transform on $H^\bullet(\mathcal{J}, \mathbb{Z})$, it also acts on $H^\bullet(\mathcal{J} \times K(\mathcal{J}), \mathbb{Z})$ by means of the Künneth decomposition. Since, by Grothendieck-Riemann-Roch, $FM^\bullet(ch(\mathcal{O}_\mathcal{I})) = ch(\mathcal{V})$, and moreover $FM^\bullet$ acts on $H^2(\mathcal{J}, \mathbb{Z})$ as an isometry $FM($ with respect to the intersection form (see [Mu 87])) we get

$$q_{2*}(q_1^*\alpha \cup c_2(\mathcal{V})) = -[q_{2*}(q_1^*\alpha \cup ch(\mathcal{V}))]_2 = -[q_{2*}(q_1^* \circ FM(\alpha) \cup ch(\mathcal{O}_\mathcal{I}))]_2$$

where the first equality is true since $\mathcal{V}$ parametrizes sheaves with trivial determinant.

But by Grothendieck-Riemann-Roch $ch_2(\mathcal{O}_\mathcal{I})$ is represented exactly by the incidence variety which can be identified, in an obvious way, to $Bl_{\mathcal{J}[2]}\mathcal{J}$: the restrictions of the two projections become then the maps b and q of the claim and then we obtain

$$[q_{2*}(q_1^* \circ FM(\alpha) \cup ch(\mathcal{O}_\mathcal{I}))]_2 = q_*(b^* \circ FM(\alpha)),$$

consequently replacing this in the previous equations we get the formula

$$p_{2*}(p_1^*\alpha \cup c_2(\mathcal{N})) = -pr^*(q_*(b^* \circ FM(\alpha)))$$

and finally by the property (19) of the Donaldson's morphism

$$f_\mathcal{N}^* \circ \varphi^* \circ \mu(\alpha) = -pr^*(q_*(b^* \circ FM(\alpha)))$$

which implies (22) since FM is in particular an integral isomorphism on $H^2(\mathcal{J}, \mathbb{Z})$.

To complete the proof of the claim it remains to relate the map $\psi : \widetilde{B}_p \to K(\widehat{\mathcal{J}})$ of Proposition 3.2.2 (see the diagram) to $pr : \mathcal{U} \to K(\mathcal{J})$.

Notice that, by Theorem 8.2.11 of [HL 97], $\varphi \circ f_N$ is constant on the fibers of pr and, since $\mathcal{U}$ is an open in a locally trivial bundle on a smooth base, it factors through the base. More precisely, using the identification $e : K(\widehat{\mathcal{J}}) \to K(\mathcal{J})$ induced by $\mathcal{J} \simeq \widehat{\mathcal{J}}$ it can be easily seen that:

$$\varphi \circ f_\mathcal{N} = i_U \circ e^{-1} \circ pr.$$

Therefore by (22)

$$pr^* \circ q_* \circ b^*(H^2(\mathcal{J}, \mathbb{Z})) = pr^* \circ (e^{-1})^* \circ i_U^* \circ \mu(H^2(\mathcal{J}, \mathbb{Z})).$$

Since the complement of $\mathcal{U}$ has complex codimension 2 in the $\mathbb{P}^1 \times \mathbb{P}^1$ bundle Q, $pr^* : H^2(K(\mathcal{J}), \mathbb{Z}) \to H^2(\mathcal{U}, \mathbb{Z})$ is injective, therefore $i_U^* \circ \mu(H^2(\mathcal{J}, \mathbb{Z})) = e^* \circ q_* \circ b^*(H^2(\mathcal{J}, \mathbb{Z}))$ and by the commutativity of the diagram in (5) of 3.2.2

$$\widetilde{i}^* \circ \widetilde{\mu}(H^2(\mathcal{J}, \mathbb{Z})) = \psi^* \circ i_U^* \circ \mu(H^2(\mathcal{J}, \mathbb{Z})) = \psi^* \circ e^* \circ q_* \circ b^*(H^2(\mathcal{J}, \mathbb{Z})).$$

$\square$

Now we finish proving Proposition 3.2.1: it is well known that $q_* \circ b^*(H^2(\mathcal{J}, \mathbb{Z}))$ is the orthogonal, with respect to the intersection form, of the submodule generated by the nodal classes in $H^2(K(\mathcal{J}), \mathbb{Z})$ [BPV 84], hence it is saturated; since $\psi : \widetilde{B}_p \to K(\widehat{\mathcal{J}})$ is a $\mathbb{P}^1$-bundle (see (4) of Proposition 3.2.2), $\psi^* \circ e^* \circ q_* \circ b^*(H^2(\mathcal{J}, \mathbb{Z}))$ is saturated too, thus the claim implies the proposition. $\square$

3.3. The divisibility of $\widetilde{\Sigma}$.

The goal of this subsection is to prove the following theorem

Theorem 3.3.1. *There exists* $A \in H^2(\widetilde{\mathcal{M}}, \mathbb{Z})$ *such that* $2A = c_1(\widetilde{\Sigma})$.

In Proposition 2.2.1 we have exhibited a birational map $b : \widetilde{\mathcal{M}} \to \widetilde{\mathbf{M}}^0_{(0,2\Theta,2)}$ furthermore we have proved that $b^*(\widetilde{\Sigma}^0_{(0,2\Theta,-2)}) = \widetilde{\Sigma}$: since $\widetilde{\mathcal{M}}$ and $\widetilde{\mathbf{M}}^0_{(0,2\Theta,2)}$ are both symplectic, they are isomorphic (via b) in codimension 1. It follows that b induces isomorphisms on Picard groups and integral 2-cohomology groups. Theorem 3.3.1 is therefore a consequence of the following proposition.

Proposition 3.3.2. $2 | c_1(\widetilde{\Sigma}^0_{(0,2\Theta,-2)})$ *in* $H^2(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}, \mathbb{Z})$.

Idea of the Proof. We will prove the desired divisibility of $c_1(\Sigma^0_{(0,2\Theta,-2)})$ using particular features of $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$. $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ is the symplectic desingularization of a moduli space $\mathbf{M}^0_{(0,2\Theta,-2)}$ whose general point parametrizes a sheaf of the form $F = (j_D)_* L$, where $j_D : D \to \mathcal{J}$ and L is a line bundle on D such that $-id^*(F) = F$ (recall that $-id^*$ fixes every $D \in |2\Theta|$) (see 3.3.7). If the support D of F does not pass through a 2-torsion point of $\mathcal{J}$, F is the pull back of a sheaf on a curve on the Kummer surface $Kum := K(\mathcal{J})$ associated to $\mathcal{J}$. This shows the existence of a dominant rational map τ from a moduli space of sheaves on Kum, $(\mathbf{M}_{(0,d^*H,-1)}(Kum)$ defined in 3.3.4) to $\mathbf{M}^0_{(0,2\Theta,-2)}$.

The map τ is studied in Proposition 3.3.6 and is proved to be generically [2:1].

The idea of the proof of the divisibility of $c_1(\widetilde{\Sigma}^0_{(0,2\Theta,-2)})$ is to extend τ to a finite map on a big (namely having complement of codimension strictly bigger than 1) open subset of $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ and then study (the closure of) its branch locus R: the general theory of double coverings implies in fact that $c_1(R)$ is 2-divisible in $H^2(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}, \mathbb{Z})$. In order to control the branch locus of the extension of τ (exhibited in Proposition 3.3.15) we need to study the complement of the image in $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ of the original map τ (see Proposition 3.3.13) and we need to recall some basic facts about the action of $\mathcal{J}[2]$ on $\mathcal{J}$ and Kum (see Remark 3.3.14). The branch locus R is studied, not determined, in the proof of the divisibility of $c_1(\widetilde{\Sigma}^0_{(0,2\Theta,-2)})$: it surely satisfies $R = \widetilde{\Sigma}^0_{(0,2\Theta,-2)} + Q$, Q being a divisor such that $2|c_1(Q) \in H^2(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}, \mathbb{Z})$. This implies the divisibility of $c_1(\widetilde{\Sigma}^0_{(0,2\Theta,-2)})$. $\square$

Thus we are reduced to prove the 2-divisibility of $c_1(\Sigma^0_{(0,2\Theta,-2)})$ in the group $H^2(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}, \mathbb{Z})$: the advantage of $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ is that it can be easily related to a moduli space of sheaves on

$$Kum := K(\mathcal{J}).$$

To explain this relation we fix notation and recall some well known classical result.

Notation 3.3.3. Let $f_{|2\Theta|} : \mathcal{J} \to |2\Theta|^\vee$ be the regular map associated with the complete linear system $|2\Theta|$.

As in the proof of Lemma 2.1.2 denote by $Kum_s := f_{|2\Theta|}(\mathcal{J})$ the singular Kummer surface associated with $\mathcal{J}$ and let $f : \mathcal{J} \to Kum_s$ be the map induced by $f_{|2\Theta|}$. It is well known that f is identified with the quotient of $\mathcal{J}$ by the involution $-id$, in particular f sends the 16 2-torsion points of $\mathcal{J}$ to the 16 singular points of Kum_s and outside of them is an unramified double covering.

Let $d : Kum \to Kum_s$ be the desingularization map, and $E := \sum_{i=1}^{16}$ be the exceptional divisor of d. The E_i's are smooth rational curves on Kum, also known as nodal curves.

Definition 3.3.4.   (1) Let H be a plane in $|2\Theta|^\vee$ and let $D := d^*H - \epsilon \sum_i E_i$ be an ample divisor in $Pic(Kum)$, we denote by $\mathbf{M}_{(0,d^*H,-1)}(Kum)$ the moduli space of sheaves on Kum having Mukai's vector $(0, d^*H, -1)$ and being semistable with respect to the polarization D.
   (2) We denote by $U_{(0,d^*H,-1)}(Kum) \subset \mathbf{M}_{(0,d^*H,-1)}(Kum)$ the open subscheme parametrizing sheaves whose Fitting subschemes (see Remark 2.1.1) do not intersect the nodal curves.

Definition 3.3.5.   (1) We denote by $V_{(0,2\Theta,-2)} := \Phi^{-1}(S \cup R(1)) \subset \mathbf{M}^0_{(0,2\Theta,-2)}$ the open subscheme parametrizing sheaves whose Fitting subscheme does not pass through 2-torsion points.

(2) We denote by $V^s \subset V_{(0,2\Theta,-2)}$ the open subscheme parametrizing stable sheaves.

We can now begin to study the relation between $\mathbf{M}_{(0,d^*H,-1)}(Kum)$ and $\mathbf{M}^0_{(0,2\Theta,-2)}$

Proposition 3.3.6. *The functor $f^* \circ d_* = b_* \circ q^* : \mathcal{C}oh(Kum) \to \mathcal{C}oh(\mathcal{J})$ (b and q as in claim 3.2.3) induces a regular map*

$$\tau : \quad U_{(0,d^*H,-1)}(Kum) \quad \to \quad \mathbf{M}^0_{(0,2\Theta,-2)}$$
$$[F] \quad \mapsto \quad [(f^* \circ d_*)F]$$

such that

    (1) $\tau(U_{(0,d^*H,-1)}(Kum)) = V_{(0,2\Theta,-2)}$
    (2) *If $x \in V^s$ then $\tau^{-1}(x)$ consists of two distinct points.*
    (3) *If $x \in V_{(0,2\Theta,-2)} \setminus V^s$ then $\tau^{-1}(x)$ consists of a point.*

Proof. Let $[F] \in U_{(0,d^*H,-1)}(Kum)$ and let $C_K \in |d^*(H)|$ be its Fitting subscheme, then $C := f^* \circ d_*(C_K)$ belongs to $\Phi^{-1}(S \cup R(1))$ (see Proposition 2.1.3). Since C_K is the quotient of C by the restriction of -1, it is easily verified that $b_* \circ q^*(F) \in \mathbf{M}^0_{(0,2\Theta,-2\eta)}$ if and only if $b_* \circ q^*(F)$ is semistable.
If $C \in S$ then the restriction of F to C_K is a line bundle and the same property holds for $f^* \circ d_*(F)$ making it a stable sheaf. If $C \in R(1)$ and the restriction of F to C_K is a line bundle (of degree 1), then $f^* \circ d_*(F)$ is a line bundle having the same degree (1) on each component of C: also in this case $f^* \circ d_*(F)$ can be proved to be stable. Finally if $C \in R(1)$ and the restriction of F to C_K is not a line bundle then F is the push-forward of a degree-0 line bundle L from the normalization (isomorphic to C^0) of C_K, it follows that $f^* \circ d_*(F)$ is the push-forward, from the desingularization of C, of a sheaf restricting to L on each connected component: it is therefore polystable. Thus τ is regular on $U_{(0,d^*H,-1)}(Kum)$.
Furthermore, for $C \in S \cup R(1)$, $\Phi^{-1}(C)$ is irreducible. In the first case, as explained in the proof of Proposition 2.1.4, $\Phi^{-1}(C)$ is identified with the kernel of the map $a : J(C) \to \mathcal{J}$ induced by the embedding $i : C \to \mathcal{J}$. To show that $Ker(a)$ is an irreducible torus it is enough to check that $i_* : H_1(C, \mathbb{Z}) \to H_1(\mathcal{J}, \mathbb{Z})$ is surjective: this follows from Lefschetz hyperplane theorem since $C \in |2\Theta|$ and Θ is ample.
In the second case, the stable locus of $\Phi^{-1}(C)$ is isomorphic to a $\mathbb{C}^*$ bundle over $\mathcal{J}$ (see the proof of Proposition 2.1.4) and it is easily seen to be dense in $\Phi^{-1}(C)$.
For $C \in S \cup R(1)$, let $M_{C_K} \subset U_{(0,d^*H,-1)}(Kum)$ be the locus parametrizing sheaves having C_K as Fitting subscheme. The restriction of τ induces a map $\tau_C : M_{C_K} \to \Phi^{-1}(C)$. τ_C is generically [2:1], in fact if F_1 and F_2 are sheaves restricting to degree-1 line bundle on C_K, then $f^* \circ d_*(F_1) \simeq f^* \circ d_*(F_2)$ if and only if they differ by tensor product with $(i_K)_*L$, where $i_K : C_K \to Kum$ is the closed embedding and L is the degree 0 line bundle on C_K associated with the unramified double covering C.
Since M_{C_K} and $\Phi^{-1}(C)$ are projective and have the same dimension τ_C is surjective: this implies item 1.
Item 2 follows since V^s parametrizes sheaves having locally free restriction to their support (see the proof of Proposition 2.1.4), hence they must be images via τ of sheaves having the same property.
Item 3 can be checked directly.

$\square$

Remark 3.3.7. Proposition 3.3.6 shows that the general sheaf F such that $[F] \in \mathbf{M}^0_{(0,2\Theta,-2)}$ is the pull-back of a sheaf on Kum_s, in particular $(-id)^*(F) = F$, since $\mathbf{M}^0_{(0,2\Theta,-2)}$ is irreducible, $\mathbf{M}^0_{(0,2\Theta,-2)}$ is in the fixed locus of the map induced on $\mathbf{M}_{(0,2\Theta,-2)}$ by $(-id)^*$

Definition 3.3.8. We define $U^s(Kum) := \tau^{-1}(V^s) \subset U_{(0,d^*H,-1)}(Kum)$ and denote by

$$\tau^s : U^s(Kum) \to V^s$$

the restriction of τ.

Remark 3.3.9. The proof of Proposition 3.3.6 shows that $U^s(Kum) \subset U_{(0,d^*H,-1)}(Kum)$ is the locus parametrizing sheaves having locally free restrictions to their supports.

The following corollary extracts from Proposition 3.3.6 what we will use to prove the divisibility of $c_1(\widetilde{\Sigma}^0_{(0,2\Theta,-2)})$.

Corollary 3.3.10. *The map* $\tau^s : U^s(Kum) \to V^s$ *is proper étale [2:1].*

Proof. Since $U^s(Kum)$ and V^s parametrize stable sheaves they are smooth and have pure dimension 6. Corollary 3.3.10 follows since, from Proposition 3.3.6, each fiber of τ^s consists of 2 distinct points. $\square$

Since the O'Grady desingularization doesn't modify the stable locus, V^s is naturally included in $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$: we now analyze $(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)} \setminus V^s)$.

Proposition 3.3.11. *Set* $\widetilde{\Phi} := \Phi \circ \widetilde{\pi}^0_{(0,2\Theta,-2)} : \widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)} \to |2\Theta|$. *For* $x \in \mathcal{J}[2]$, *let* H_x *be the plane in* $|2\Theta|$ *corresponding to curves passing through* x, *then the locus* $\Delta_x := \widetilde{\Phi}^{-1}H_x \subset \widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ *is an irreducible divisor.*

This proposition is a consequence of the following lemma

Lemma 3.3.12. *If* $C \in N(1)$ *then* $\Phi^{-1}(C)$ *is irreducible.*

Proof. As explained in the proof of Proposition 2.1.4 the open dense subset of $\Phi^{-1}(C)$, whose points parametrize sheaves restricting to line bundles over C, is a $\mathbb{C}^*$ bundle over the kernel of the map $a: J(\widetilde{C}) \to \mathcal{J}$ induced on Albanese varieties by the map $h : \widetilde{C} \to \mathcal{J}$ obtained composing the normalization map of C with its inclusion in $\mathcal{J}$.
The irreducibility of $\Phi^{-1}(C)$ is again a consequence of the surjectivity of $h_* : H_1(\widetilde{C}, \mathbb{Z}) \to H_1(\mathcal{J}, \mathbb{Z})$: since it can be checked that the natural image of $\widetilde{C}$ in the blow up $Bl_x\mathcal{J}$ of $\mathcal{J}$ in x is an ample divisor and moreover the blow up map induces an isomorphism on integral 1-homology groups, the desired surjectivity follows again by Lefschetz hyperplane theorem. $\square$

Proof of Proposition 3.3.11. Since $H_x \cap (S \cup R(1)) = \emptyset$ we have,

$$\widetilde{\Phi}^{-1}H_x = \widetilde{\Phi}^{-1}(H_x \cap N(1)) \cup \widetilde{\Phi}^{-1}(H_x \cap (N(2) \cup N(3) \cup R(2) \cup D)).$$

But $\widetilde{\Phi}^{-1}(H_x \cap N(1)) = \Phi^{-1}(H_x \cap N(1))$ and by Lemma 3.3.12 the latter is a fibration with irreducible 3 dimensional fiber and irreducible 2 dimensional base: hence it is a codimension 1 locally closed subset.
On the other hand $\widetilde{\Phi}^{-1}(H_x \cap (N(2) \cup N(3) \cup R(2) \cup D))$ is the inverse image of a codimension 2 subset via a Lagrangian fibration on an irreducible symplectic variety: by Theorem 2 (step 5) of [Ma 99] it cannot contain a codimension 1 subset. It follows that the divisor $\Delta_x := \widetilde{\Phi}^{-1}H_x$ is irreducible. $\square$

Finally we determine the multiplicity of $\widetilde{\Phi}^*H_x$

Proposition 3.3.13.

$$\widetilde{\Phi}^*H_x = \Delta_x.$$

Proof. Let $H \subset |2\Theta|$ be a generic plane: we will show that $c_1(\widetilde{\Phi}^*H) = c_1(\widetilde{\Phi}^*H_x)$ is indivisible in $H^2(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}, \mathbb{Z})$. Since $\widetilde{\Phi}^*H$ is a reduced irreducible divisor we will prove the proposition exhibiting a curve intersecting it transversally in a unique point. Consider the curves in $|2\Theta|$ passing through 4 fixed points $\{a, -a, b, -b\}$ outside $\mathcal{J}[2]$: they form line L in $|2\Theta|$, this line consists of the inverse images of the hyperplane sections of the singular Kummer surface passing through 2 fixed points. It is easy to produce a flat family $\mathcal{F}$ on $\mathcal{J} \times L$ such that, letting $j : C \to \mathcal{J}$ be the inclusion of $C \in |2\Theta|$, for all $C \in L$

$$\mathcal{F}_{|\mathcal{J} \times C} = j_*\mathcal{O}(j^{-1}(a) + j^{-1}(-a)).$$

This family induces a modular map $f_\mathcal{F}$ such that $\Phi \circ f_\mathcal{F}$ is the identity map. For H general, the image of $f_\mathcal{F}$ obviously intersects $\Phi^{-1}H$ transversally in a unique point: it follows that also the image of the lifting of $f_\mathcal{F}$ to $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ intersects $\widetilde{\Phi}^{-1}H$ transversally in a unique point. $\qquad \square$

We are now going to prove that $c_1(\widetilde{\Sigma}^0_{(0,2\Theta,-2)}) \in H^2(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}, \mathbb{Z})$ is a class divisible by 2: it will follow from the existence of a square root of the class $[\widetilde{\Sigma}^0_{(0,2\Theta,-2)}]$ in $Pic(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)})$.
We will give a partial completion of the étale double covering of Corollary 3.3.10 and studying its branch locus we will obtain the existence of such a square root: τ has to be completed since its image does not contain the divisors Δ_x and $\widetilde{\Sigma}^0_{(0,2\Theta,-2)}$ and therefore it cannot give relations on $Pic(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)})$.
We anticipate that we will not be able to say whether our new double covering ramifies or not on the Δ_x's, but it will be enough to state that if it ramifies on a Δ_x it ramifies on each Δ_x: this will be achieved by means of a $\mathcal{J}[2]$-action, on both the domain and the codomain, which permutes the Δ_x and is compatible with the double covering.
In the following remark we define and explain the actions that we will use.

Remark 3.3.14. (1) Let x be a 2-torsion point of $\mathcal{J}$, then obviously

$$(-id) \circ t_x = t_x \circ (-id) : \mathcal{J} \to \mathcal{J}$$

and this implies that the action by translation of $\mathcal{J}[2]$ on $\mathcal{J}$ descends to an action on the quotient Kum_s. Finally this action lifts to the blow up Kum of the singular locus of Kum_s.

(2) The $\mathcal{J}[2]$ action on $\mathcal{J}$ induces $\mathcal{J}[2]$ actions on $\mathbf{M}^0_{(0,2\Theta,-2)}$, on $\Sigma^0_{(0,2\Theta,-2)}$ and on $\Omega^0_{(0,2\Theta,-2)}$. In fact $\mathcal{J}[2]$ obviously acts on $\mathbf{M}_{(0,2\Theta,-2)}$, on $\Sigma_{(0,2\Theta,-2)}$, and on $\Omega_{(0,2\Theta,-2)}$; furthermore, since for $x \in \mathcal{J}[2]$ $t_x^*2\Theta = 2\Theta$ and for $[F] \in \mathbf{M}_{(0,2\Theta,-2)}$ $\sum c_2(t_x^*(F)) = deg(c_2(F))x + \sum c_2(F)$, $\mathbf{M}^0_{(0,2\Theta,-2)}$ is invariant for this action.
Moreover since $\mathcal{J}[2]$ permutes the 2-torsion points on $\mathcal{J}$, the open subscheme $V_{(0,2\Theta,-2)}$ defined in 3.3.5 is invariant under the $\mathcal{J}[2]$ action and, since translations on $\mathcal{J}$ do not change stability, the open subscheme $V^s \subset V_{(0,2\Theta,-2)}$ also defined in 3.3.5 is invariant too.

(3) The $\mathcal{J}[2]$ action on Kum induces an action on $\mathbf{M}_{(0,d^*H,-1)}(Kum)$. In fact, since we choose a polarization $D = d^*H - \epsilon \sum E_i$ (see 3.3.4) symmetric with respect to this action, the pull-backs by automorphisms induced by $\mathcal{J}[2]$ do not change stability and semistability. Moreover, since the $\mathcal{J}[2]$ action on Kum_s permutes the singular points, the open subscheme $U_{(0,d^*H,-1)}(Kum)$ defined in 3.3.4 is invariant under this action and, since the pull back of a sheaf, locally free on its support, has the same property, the open subscheme $U^s(Kum) \subset U_{(0,d^*H,1)}(Kum)$ defined in 3.3.8 is invariant too .

(4) Since by 1) of this remark the actions of $\mathcal{J}[2]$ commute with $f : \mathcal{J} \to Kum_s$ and $d : Kum_s \to Kum$, the pull backs of sheaves via the automorphisms induced by a fixed

$x \in \mathcal{J}[2]$ on $\mathcal{J}$ and Kum commute with $f_* \circ d^*$, therefore the regular morphism

$$\tau : U_{(0,d^*H,-1)}(Kum) \to \mathbf{M}^0_{(0,2\Theta,-2)}$$

(defined in Proposition 3.3.6) commutes with the $\mathcal{J}[2]$-actions on $U_{(0,d^*H,-1)}(Kum)$ and $\mathbf{M}^0_{(0,2\Theta,-2)}$ described in 2) and 3) of this remark.

Moreover, since by 2) and 4) the open subvarieties $V^s \subset \mathbf{M}^0_{(0,2\Theta,-2)}$ and $U^s(Kum) := \tau^{-1}(V^s)$ are $\mathcal{J}[2]$ invariant, the restriction of τ

$$\tau^s : U^s(Kum) \to V^s$$

defined in 3.3.8 commutes with the induced actions on $U^s(Kum)$ and V^s.

We are now ready to describe the desired extension of τ^s.

Proposition 3.3.15.   (1) *Let $j : V^s \to \widetilde{V} := \widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)} \setminus \widetilde{\Omega}_{(0,2\Theta,-2)}$ be the natural inclusion (recall from the definition 3.3.5 that V^s is a subset of the stable locus of $\mathbf{M}^0_{(0,2\Theta,-2)}$), then the $\mathcal{J}[2]$-action on V^s extends to $\widetilde{V}$.*

(2) *There exist a smooth algebraic variety $\widetilde{U}(Kum)$, an open embedding*

$$i : U^s(Kum) \to \widetilde{U}(Kum),$$

an action of $\mathcal{J}[2]$ on $\widetilde{U}(Kum)$ extending the given $\mathcal{J}[2]$-action on $U^s(Kum)$ and a proper map $\widetilde{\tau} : \widetilde{U}(Kum) \to \widetilde{V}$ making commutative the following diagram:

$$
\begin{array}{ccc}
U^s(Kum) & \xrightarrow{\ i\ } & \widetilde{U}(Kum) \\
{\scriptstyle \tau^s}\downarrow & & \downarrow{\scriptstyle \widetilde{\tau}} \\
V^s & \xrightarrow{\ j\ } & \widetilde{V}.
\end{array}
$$

In particular $\widetilde{\tau}$ commutes with the $\mathcal{J}[2]$ actions on $\widetilde{U}(Kum)$ and $\widetilde{V}$.

Proof. (1): Since by O'Grady's construction $\widetilde{V}$ is simply the blow up of $\mathbf{M}^0_{(0,2\Theta,-2)} \setminus \Omega^0_{(0,2\Theta,-2)}$ along $\Sigma^0_{(0,2\Theta,-2)} \setminus \Omega^0_{(0,2\Theta,-2)}$ (see Remark 1.1.6), item 1) follows from the $\mathcal{J}[2]$ invariance of $\Sigma^0_{(0,2\Theta,-2)}$ and $\Omega^0_{(0,2\Theta,-2)}$ proved in 2) of remark 3.3.14.

(2): By (1) there is a $\mathcal{J}[2]$ equivariant diagram

(23)
$$
\begin{array}{ccc}
U^s(Kum) & \xrightarrow{\ id\ } & U^s(Kum) \\
{\scriptstyle \tau^s}\downarrow & & \downarrow{\scriptstyle \tau^1} \\
V^s & \xrightarrow{\ j\ } & \widetilde{V}
\end{array}
$$

where obviously the horizontal maps are open embeddings and $\tau^1 := j \circ \tau^s$.

To replace τ^1 with a proper map let $\Gamma \subset U^s(Kum) \times \widetilde{V}$ be the graph of τ^1, consider the inclusion $U^s(Kum) \subset \mathbf{M}_{(0,d^*H,-1)}(Kum)$ and let $\overline{\Gamma(Kum)} \subset \mathbf{M}_{(0,d^*H,-1)}(Kum) \times \widetilde{V}$ be the closure of $\Gamma(Kum)$. Let $\tau^2 : \overline{\Gamma(Kum)} \to \widetilde{V}$ be the restriction of the projection $p_2 : \mathbf{M}_{(0,d^*H,-1)}(Kum) \times \widetilde{V} \to \widetilde{V}$ and $i_{\overline{\Gamma}} : U^s(Kum) \to \overline{\Gamma(Kum)}$ be the natural open embedding: then, clearly, the following diagram is commutative:

(24)
$$
\begin{array}{ccc}
U^s(Kum) & \xrightarrow{\ i_{\overline{\Gamma}}\ } & \overline{\Gamma(Kum)} \\
{\scriptstyle \tau^1}\downarrow & & \downarrow{\scriptstyle \tau^2} \\
\widetilde{V} & \xrightarrow{\ id\ } & \widetilde{V}
\end{array}
$$

τ^2 is proper since it is the restriction to a closed subvariety of the projection p_2 which is proper since $\mathbf{M}_{(0,d^*H,-1)}(Kum)$ is projective.

Moreover the $\mathcal{J}[2]$ action on $U^s(Kum)$ extends via $i_{\overline{\Gamma}}$ to $\overline{\Gamma(Kum)}$.

To prove this notice that, since τ^1 is $\mathcal{J}[2]$ equivariant, the $\mathcal{J}[2]$-action on $U^s(Kum)$ can be identified with the restriction to $\Gamma(Kum)$ of the $\mathcal{J}[2]$ diagonal action on $U^s(Kum) \times \widetilde{V}$. Furthermore the $\mathcal{J}[2]$ action on $U^s(Kum) \times \widetilde{V}$ is in its turn, by (3) of remark 3.3.14, the restriction of a diagonal action on $\mathbf{M}_{(0,d^*H,-1)}(Kum) \times \widetilde{V}$. Finally, being $\Gamma(Kum)$ invariant with respect to this action, its closure $\overline{\Gamma(Kum)}$ is invariant too, and the restriction of the $\mathcal{J}[2]$-action on $\mathbf{M}_{(0,d^*H,-1)}(Kum) \times \widetilde{V}$ to $\overline{\Gamma(Kum)}$ provides the desired extension.

By a general result, see [AW 97, BM 97], for any finite group acting on an algebraic variety there always exists in characteristic 0 an equivariant resolution of singularities: let then $r : \widetilde{U}(Kum) \to \overline{\Gamma(Kum)}$ be such a resolution for the $\mathcal{J}[2]$ action on $\overline{\Gamma(Kum)}$. Since $i_{\overline{\Gamma}}$ is an open embedding of a smooth variety, it lifts to an open embedding $i_{\widetilde{U}^s(Kum)} : U^s(Kum) \to \widetilde{U}(Kum)$ and setting $\widetilde{\tau} := \tau^2 \circ r$ we get a $\mathcal{J}[2]$ equivariant commutative diagram

(25)
$$
\begin{array}{ccc}
U^s(Kum) & \xrightarrow{i_{\widetilde{U}^s(Kum)}} & \widetilde{U}(Kum) \\
\tau^1 \downarrow & & \widetilde{\tau} \downarrow \\
\widetilde{V} & \xrightarrow{\quad id \quad} & \widetilde{V}
\end{array}
$$

And joining this diagram with (23) we complete the proof Proposition 3.3.15. $\qquad\square$

Remark 3.3.16. Since by Corollary 3.3.10 τ^s is proper, we have $\tau^s(\widetilde{U}(Kum) \setminus U^s(Kum)) \subset (\widetilde{V} \setminus V^s)$, in particular the image of the contracted locus does not intersect V^s.

Finally we get the divisibility of $c_1(\widetilde{\Sigma}^0_{(0,2\Theta,-2)})$.

Proof of Proposition 3.3.2. Let $C \subset \widetilde{U}(Kum)$ be the subvariety contracted by $\widetilde{\tau}$, namely

$$C := \{x \in \widetilde{V} : \ dim(\widetilde{\tau}^{-1}(\widetilde{\tau}(x))) > 0\}$$

set $\widetilde{V}^0 := \widetilde{V} \setminus \widetilde{\tau}(D)$ and $\widetilde{U}^0(Kum) := \widetilde{\tau}^{-1}(\widetilde{V}^0)$, then $\widetilde{V}^0$ and $\widetilde{U}^0(Kum)$ are obviously $\mathcal{J}[2]$ invariant and the restriction

$$\widetilde{\tau}^0 : \widetilde{U}^0(Kum) \to \widetilde{V}^0$$

is $\mathcal{J}[2]$ equivariant. Furthermore, by construction, $\widetilde{\tau}^0$ is a proper map with finite fibers , hence it is finite, since $\widetilde{U}^0(Kum)$ and $\widetilde{V}^0$ are smooth we deduce that $\widetilde{\tau}^0$ is flat.

The theory of double coverings can be applied to deduce that the class in $Pic(\widetilde{V}^0)$ of the branch locus of $\widetilde{\tau}^0$ has a square root. Since $codim(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)} \setminus \widetilde{V}^0, \widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}) > 1$, the open embedding $\widetilde{V}^0 \subset \widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ induces an isomorphism $Pic(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}) \simeq Pic(\widetilde{V}^0)$ which will give the existence in $Pic(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)})$ of a square root of the line bundle associated to the closure in $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ of the branch locus of $\widetilde{\tau}^0$.

By the commutativity of diagram in (2) of Proposition 3.3.15, since τ^s is everywhere [2:1] by Corollary 3.3.10, the ramification locus R of $\widetilde{\tau}^0$ and obviously its closure $\overline{R}$ in $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ are included in $\widetilde{\Sigma}^0_{(0,2\Theta,-2)} \bigcup_{x \in \mathcal{J}[2]} \Delta_x$. Furthermore by the $\mathcal{J}[2]$ equivariance of $\widetilde{\tau}^0$ and since $\mathcal{J}[2]$ acts transitively on the Δ_x, as preannounced, if there is an $x \in \mathcal{J}[2]$ such that $\Delta_x \subset \overline{R}$ then $\bigcup_{x \in \mathcal{J}[2]} \Delta_x \subset \overline{R}$.

Therefore there are only four possible cases:

(1) $\overline{R} = \emptyset$,

(2) $\overline{R} = \bigcup_{x \in \mathcal{J}[2]} \Delta_x$,

(3) $\overline{R} = \widetilde{\Sigma}^0_{(0,2\Theta,-2)} \bigcup_{x \in \mathcal{J}[2]} \Delta_x$,

(4) $\overline{R} = \widetilde{\Sigma}^0_{(0,2\Theta,-2)}$.

The first 2 cases are not possible.

Suppose by absurd the contrary: then, denoting by $\overline{R}$ also the associated reduced divisor, by Proposition 3.3.13 we would have the following equality of divisors

$$\overline{R} = \widetilde{\Phi}^* D$$

where $D = 0$ if $\overline{R} = \emptyset$, or $D = \sum_{x \in \mathcal{J}[2]} H_x$ if $\overline{R} = \bigcup \Delta_x$.

In both of these cases there would exist a divisor $D^1 \in Div(|2\Theta|^\vee)$ such that $[2D^1] = [D]$ in $Pic(|2\Theta|^\vee)$ and since $Pic(\widetilde{V}^0) = Pic(\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)})$ is free, the restriction of the class $[\widetilde{\Phi}^* D^1]$ to $\widetilde{V}^0$ would be the unique square root of the class $[R] \in Pic(\widetilde{V}^0)$. Moreover since the codimension of the complement of $\widetilde{V}^0$ in $\widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)}$ is bigger than 1, there would exist a unique (up to scalars) regular section σ of $\mathcal{O}_{\widetilde{V}^0}(R)$ vanishing with multiplicity 1 on each component of R and obviously it would be the restriction to $\widetilde{V}^0$ of the pull back via $\widetilde{\Phi}$ of a section s of $\mathcal{O}_{|2\Theta|^\vee}(D)$. Therefore the double covering $\widetilde{\tau}^0$ would be the one defined by means of $[\widetilde{\Phi}^* D]$ and $\widetilde{\Phi}^* s$, in particular its restriction to a fiber of $\widetilde{\Phi}$ over a p not in D would yield a double covering defined by means of the trivial square root of the trivial line bundle and of a constant section, namely we would get the trivial double covering: this is absurd because for $C \in |2\Theta|$ smooth, $(\tau^s)^{-1}(\Phi^{-1}(C))$ is identified with $M_{C_K} \simeq Pic^1(C_K)$, where $C_K = f(C)$ is a smooth genus 3 curve (see the notation of the proof of Proposition 3.3.6).

Since by Proposition 3.3.13 $\mathcal{O}(\sum_{\mathcal{J}[2]} \Delta_x) = \widetilde{\Phi}^* \mathcal{O}(16)$, both $[\overline{R}]$ and $[\sum_{\mathcal{J}[2]} \Delta_x]$ have a square root, therefore in the last 2 cases $[\widetilde{\Sigma}^0_{(0,2\Theta,-2)}]$ has a square root too. $\square$

3.4. The integral basis.

We can now present a basis for $H^2(\widetilde{M}, \mathbb{Z})$.

Theorem 3.4.1. *Let $\{\alpha_i\}_{i=1}^6$ be a basis for $H^2(\mathcal{J}, \mathbb{Z})$, then $\{\mu(\alpha_i), c_1(\widetilde{B}), A\}$ is a basis for $H^2(\widetilde{M}, \mathbb{Z})$.*

Proof. By Poincaré duality, since we already know $rk(H^2(\widetilde{\mathcal{M}}, \mathbb{Z})) = 8$, it is enough to show there are 8 elements in $H_2(\widetilde{\mathcal{M}}, \mathbb{Z})$ such that the determinant of the evaluation matrix of these two 8-tuples is 1. By Proposition 3.2.1 and Poincaré duality we can find $\{\alpha_i^*\}_{i=1}^6$ such that $det(<\widetilde{\mu}(\alpha_i), \alpha_j^*>) = 1$. Let γ and δ be as in the proof of Proposition (7.3.3) of in [OG 03], then, as shown there we have

$$\begin{pmatrix} <A, \delta> & <A, \gamma> \\ <c_1(\widetilde{B}), \delta> & <c_1(\widetilde{B}), \gamma> \end{pmatrix} = \begin{pmatrix} <\frac{c_1(\widetilde{\Sigma})}{2}, \delta> & <\frac{c_1(\widetilde{\Sigma})}{2}, \gamma> \\ <c_1(\widetilde{B}), \delta> & <c_1(\widetilde{B}), \gamma> \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}.$$

The second equality is proved in Proposition (7.3.3) of in [OG 03]. Furthermore, since δ and γ are contracted by $\varphi \circ \widetilde{\pi}$, we have $<\widetilde{\mu}(\alpha_i), \delta> = <\widetilde{\mu}(\alpha_i), \gamma> = 0$ for each i: so the determinant of the whole intersection matrix is given by

$$det(<\widetilde{\mu}(\alpha_i), \alpha_j^*>) \cdot det\begin{pmatrix} <A, \delta> & <A, \gamma> \\ <c_1(\widetilde{B}), \delta> & <c_1(\widetilde{B}), \gamma> \end{pmatrix} = 1$$

$\square$

3.5. **Explicit computation.** We now compute the Beauville form of $\widetilde{\mathcal{M}}$ in terms of the given basis of $H^2(\widetilde{\mathcal{M}}, \mathbb{Z})$.

The final result is the following

Theorem 3.5.1. *Set* $\Lambda := \mathbb{Z}A \oplus \mathbb{Z}c_1(\widetilde{B}) \subset H^2(\widetilde{\mathcal{M}}, \mathbb{Z})$. *There is a direct sum decomposition*

$$H^2(\widetilde{\mathcal{M}}, \mathbb{Z}) = \widetilde{\mu}(H^2(\mathcal{J}, \mathbb{Z})) \oplus_\perp \Lambda$$

orthogonal with respect to $B_{\widetilde{\mathcal{M}}}$.

The map $\widetilde{\mu} : (H^2(\mathcal{J}, \mathbb{Z}), (,)_{\mathcal{J}}) \longrightarrow (H^2(\widetilde{\mathcal{M}}, \mathbb{Z}), B_{\widetilde{\mathcal{M}}})$ *is an isometric embedding.*
Furthermore the matrix of the Beauville's form on Λ *is given the following formula:*

	A	$c_1(\widetilde{B})$
A	-2	2
$c_1(\widetilde{B})$	2	$-4.$

Finally the Fujiki constant of $\widetilde{\mathcal{M}}$ *is* $c_{\widetilde{\mathcal{M}}} = 60$.

We will prove this theorem at the end of this section, after having computed the necessary intersection numbers on $\widetilde{\mathcal{M}}$.

Few intersection numbers are actually needed to completely determine the Beauville's form on $\widetilde{\mathcal{M}}$: most of them, those involving $\widetilde{B}$ and $\widetilde{\Sigma}$, can be computed in terms of the known geometry of explicit subvarieties ($\widetilde{B}$, $\widetilde{\Sigma}$ and $\widetilde{B} \cap \widetilde{\Sigma}$) of $\widetilde{M}$.

Nevertheless to obtain the right normalization we will need to compute also at least a non zero sextuple self-intersection of a class in $H^2(\widetilde{M})$. More precisely we want to prove the following proposition.

Proposition 3.5.2.

$$(26) \qquad \int_{\widetilde{M}} \widetilde{\mu}(\omega + \overline{\omega})^6 = 60 \left(\int_{\mathcal{J}} (\omega + \overline{\omega})^2 \right)^3.$$

This self intersection can be related to a known intersection number on $Hilb^3(Kum)$. The existence of a relation is provided by the following proposition.

Proposition 3.5.3. *There exists a generically injective rational map*

$$\beta : Hilb^3(Kum) \dashrightarrow \mathbf{M}_{(0,d^*H,-1)}.$$

Proof. β is given as follows. Given 3 general points on Kum there exists a unique curve C in $|d^*H|$ passing through them. Considering the push forward on Kum of the ideal of the 3 points on C and tensoring it with $\mathcal{O}(d^*H)$, we get a sheaf in $\mathbf{M}_{(0,d^*H,-1)}$. β is easily seen to be generically injective (see [De 99, Be 99]). $\square$

Therefore we can relate $Hilb^3(Kum)$ with $\widetilde{\mathcal{M}}$ by means of the rational generically [2:1] map

$$(27) \qquad h := \widetilde{\pi}^{-1} \circ (t^0)^{-1} \circ (fm^0)^{-1} \circ \tau \circ \beta : Hilb^3(Kum) \dashrightarrow \widetilde{\mathbf{M}}^0_{(0,2\Theta,-2)},$$

where $fm^0 : \mathbf{M}^0_{(2,2\Theta,0)} \to \mathbf{M}^0_{(0,2\Theta,-2)}$ and $t^0 : \mathbf{M}^0_{(2,0,-2)} \to \mathbf{M}^0_{(2,2\Theta,0)}$ are the restrictions of the map fm and t defined in the proof of Proposition 2.2.1.

In order to compute $(\widetilde{\mu}(\omega + \overline{\omega}))^6$ we are going to identify $h^* \circ \widetilde{\mu}(\omega)$. A great simplification in this identification is provided by the following.

Remark 3.5.4. The class $\widetilde{\mu}(\omega) \in H^2(\mathcal{M}, \mathbb{Z})$ has a holomorphic representative: this is a particular instance of the last remark on page 504 of [OG 03] asserting that $\widetilde{\mu}$ is a morphism of Hodge structures.

Since h is a rational map between smooth varieties it is defined in codimension 1, by the previous remark $h^* \circ \widetilde{\mu}(\omega)$ is, where defined, an holomorphic form, therefore by Hartog's theorem it extends to the whole $Hilb^3(Kum)$. We want to compare this extension that we will still call $h^* \circ \widetilde{\mu}(\omega)$ with another holomorphic form defined by means of ω on $Hilb^3(Kum)$. Precisely, letting ω_K be the unique form on Kum such that its pull back to $\mathcal{J}$ via the rational [2:1] map extends to ω, we want to compare $h^* \circ \widetilde{\mu}(\omega)$ with the holomorphic 2-form $\vartheta(\omega_K)$ determined by requiring that its pull-back to Kum^3 (via the natural rational map) is equal to $\sum_{i=1}^3 \pi_i^* \omega_K$ (π_i being the i-th projection).

Theorem 3.5.5.

$$(28) \qquad\qquad\qquad h^* \circ \widetilde{\mu}(\omega) = 2\vartheta(\omega_K)$$

We need a lemma.

Lemma 3.5.6. *There is a Zarisky open subset $U(Kum) \subset Hilb^3(Kum)$ such that the following equality of cohomology classes on $U(Kum)$ holds*

$$(h^* \circ \widetilde{\mu}(\omega))_{|U(Kum)} = 2\vartheta(\omega_K)_{|U(Kum)}.$$

Proof. The first step in proving Lemma 3.5.6 consists in showing that there exists an open subscheme $U(Kum) \subset Hilb^3(Kum)$ such that h is defined on $U(Kum)$, $h(U(Kum))$ is included in the stable locus of $\mathbf{M}^0_{(2,0,-2)}$ and moreover the restriction

$$h_U : U(Kum) \to h(U(Kum)) \subset \widetilde{\mathbf{M}}^0_{(2,0,-2)}$$

is identified to a modular map induced by a flat family $\mathcal{F}$ on $\mathcal{J} \times U(Kum)$.

The open subscheme $U(Kum)$ is simply the locus $U(Kum) \subset Hilb^3(Kum)$ parametrizing sheaves having a good behavior with respect to the functors used in the definition of the map h. Precisely $U(Kum)$ parametrizes sheaves of ideals I_Z, where $Z = \{z_1, z_2, z_3\}$ consists of 3 distinct points such that:

(1) Exists a unique $C_K \in |d^*H|$ such that $z_i \in C_K$ and moreover C_K is smooth.
(2) Letting $j : C_K \to Kum$ be the closed embedding and setting $p_i := j^{-1}(z_i)$, the sheaf $f^* \circ d_*(j_*(\mathcal{O}_{C_K}(-\sum p_i)) \otimes \mathcal{O}(d^*H))$ satisfies W.I.T. (index 1) and its Fourier-Mukai transform is stable.

The family $\mathcal{F}$ is defined as follows.
Letting $\mathcal{I}$ be a tautological family of ideals on $Kum \times U(Kum)$ and letting p_K and q_K be the projections of $Kum \times U(Kum)$, by 1) $q_{K*}\mathcal{H}om(p_K^*\mathcal{O}_{Kum}(-d^*H), \mathcal{I})$ is a line bundle L on $U(Kum)$: the line bundle $q_{K*}\mathcal{H}om(p_K^*\mathcal{O}_{Kum}(-d^*H) \otimes q_K^*L), \mathcal{I})$ has then a nowhere vanishing section that induces an exact sequence:

$$(29) \qquad\qquad 0 \to p_K^*(\mathcal{O}_{Kum}(-d^*H)) \otimes q_K^*L \to \mathcal{I} \to \mathcal{G} \to 0$$

defining a family $\mathcal{G}$ such that $\mathcal{G}_{|Kum \times Z} \simeq j_*(\mathcal{O}_{C_K}(-\sum p_i))$.
Setting $\mathcal{T} := (b \times id)_* \circ (q \times id)^* : Coh(Kum \times U(Kum)) \to Coh(\mathcal{J} \times U(Kum))$ we can define a new family on $\mathcal{J} \times U(Kum)$

$$(30) \qquad\qquad\qquad \mathcal{H} := \mathcal{T}(\mathcal{G} \otimes p_K^*(\mathcal{O}(d^*H))).$$

Finally, letting p_J and q_J be the projections of $\mathcal{J} \times U(Kum)$ and denoting $\mathcal{FM}_U$ the Fourier-Mukai transform for families of sheaves on $\mathcal{J}$ parametrized by $U(Kum)$, using 2) we can define the family $\mathcal{F}$

$$(31) \qquad \mathcal{F} := p_J^*(\mathcal{O}(-\Theta)) \otimes \mathcal{FM}_U(\mathcal{H}).$$

and by constrction, the modular map h_U induced by $\mathcal{F}$ is identified on $U(Kum)$ with h. We can now use the property (19) to compare $\vartheta(\omega)_{|U(Kum)}$ and $h^* \circ \widetilde{\mu}(\omega)_{|U(Kum)}$. Using (19) and the orthogonality of $c_1(\Theta)$ and ω:

$$(32) \qquad h^* \circ \widetilde{\mu}(\omega)_{|U(Kum)} = h_U^* \circ \varphi^* \circ \mu(\omega) = q_{J*}(p_J^*(\omega) \cup c_2(\mathcal{F}))$$

$$= q_{J*}(p_J^*(\omega) \cup c_2(\mathcal{FM}_U(\mathcal{H}))) = -[q_{J*}(p_J^*(\omega) \cup ch(\mathcal{FM}_U(\mathcal{H})))]_2$$

(the last equality holds since $\mathcal{FM}_U(\mathcal{H})$ parametrizes sheaves with fixed determinant). Decomposing $ch(\mathcal{FM}_U(\mathcal{H}))$ by means of the Künneth decomposition of $H^\bullet(\mathcal{J} \times U(Kum))$ and the Hodge decomposition of $H^\bullet(\mathcal{J})$ it is easily seen that the unique component giving non zero contribution in the last term of (32) is the one in $H^{0,2}(\mathcal{J}) \otimes H^2(U(Kum))$. On the other hand using Grothendieck-Riemann-Roch

$$(33) \qquad ch(\mathcal{FM}_U(\mathcal{H})) = FM_U(ch(\mathcal{H}))$$

where FM_U is the automorphism of $H^\bullet(\mathcal{J} \times U(Kum))$ induced by means of the Künneth decomposition of $H^\bullet(\mathcal{J} \times U(Kum))$ and the automorphism FM induced by $\mathcal{FM}$ on $H^\bullet(\mathcal{J})$ (see [Mu 87]). Since $FM(H^2(\mathcal{J})) = H^2(\mathcal{J})$ and FM, being defined by means of an algebraic cycle, respects the Hodge decomposition we get $FM(H^{0,2}(\mathcal{J})) = H^{0,2}(\mathcal{J})$ and since $FM \circ FM = -id^*$, we also see that FM is the identity on $H^{0,2}(\mathcal{J})$.
It follows that the component in $H^{0,2}(\mathcal{J}) \otimes H^2(U(Kum))$ of $ch(\mathcal{FM}_U(\mathcal{H}))$ is the same as the one of $ch(\mathcal{H})$. Therefore

$$(34) \qquad [q_{J*}(p_J^*(\omega) \cup ch(\mathcal{FM}_U(\mathcal{H})))]_2 = [q_{J*}(p_J^*(\omega) \cup ch(\mathcal{H}))]_2$$

We want now to compare $q_{J*}(p_J^*(\omega) \cup ch_2(\mathcal{H}))$ and $q_{K*}(p_K^*(\omega_K) \cup ch_2(\mathcal{G} \otimes p_K^*(\mathcal{O}(d^*H))))$ recalling by the definition (30) that

$$\mathcal{T}(\mathcal{G} \otimes p_K^*(\mathcal{O}(d^*H))) = (b \times id)_* \circ (q \times id)^*((\mathcal{G} \otimes p_K^*(\mathcal{O}(d^*H)))) = \mathcal{H}.$$

Letting the algebraic cycle $D := \sum_i n_i D_i$ be a representative of $ch_2((\mathcal{G} \otimes p_K^*(\mathcal{O}(-d^*H))))$ not intersecting $\bigcup_{x \in \mathcal{J}[2]} E_x \times U(Kum)$ and setting $B_i := (b \times id)_* \circ (q \times id)^*(D_i)$, it is easily seen that there exist étale double coverings $g_i : B_i \to D_i$ such that $g_i^*(p_K^*(\omega_K)_{|D_i}) = (p_J^*(\omega))_{|B_i}$: since $\sum_i B_i$ is a representative of $ch_2(\mathcal{H})$ we get

$$(35) \qquad q_{J*}(p_J^*(\omega) \cup ch_2(\mathcal{H})) = \sum_{i=1}^n n_i q_{J*}((p_J^*(\omega))_{|B_i}) =$$

$$2 \sum_{i=1}^n n_i q_{K*}((p_K^*(\omega_K))_{|D_i}) = 2q_{K*}(p_K^*(\omega_K) \cup ch_2(\mathcal{G} \otimes p_K^*(\mathcal{O}(d^*H)))).$$

Finally using the orthogonality of ω_K and $d^*(H)$ and applying Whitney's formula to the sequence (29) a straightforward computation yields

$$(36) \qquad 2q_{K*}(p_K^*(\omega_K) \cup ch_2(\mathcal{G} \otimes p_K^*(\mathcal{O}(d^*H)))) = 2q_{K*}(p_K^*(\omega_K) \cup ch_2(\mathcal{G})) =$$

$$2q_{K*}(p_K^*(\omega_K) \cup (ch_2(\mathcal{I}) - ch_2(p_K^*(\mathcal{O}(-d^*H)) \otimes q_K^*L))) = 2q_{K*}(p_K^*(\omega_K) \cup ch_2(\mathcal{I})) =$$

$$-2\vartheta(\omega_K)_{|U(Kum)}.$$

The last equality follows from the known description of the 2-cohomology of $Hilb^3(Kum)$ (see [OG 01]). Joining (36) with the equations (32), (34), (35) and we get the Lemma. $\qquad\square$

Proof of Theorem 3.5.5. In general, if X is a projective variety and $U \subset X$ is a Zarisky open subset, then the natural map from $H^0(\Omega_X^2)$ to $H^2(U, \mathbb{C})$ is injective. Since by Lemma 3.5.6 $h^* \circ \widetilde{\mu}(\omega)_{|U} = 2\vartheta(\omega_K)_{|U}$, it follows that $h^* \circ \widetilde{\mu}(\omega) = 2\vartheta(\omega_K)$. $\qquad\square$

As an immediate consequence we can compute $\int_{\widetilde{\mathcal{M}}} \widetilde{\mu}(\omega + \overline{\omega})^6$ and, hence, prove Proposition 3.5.2.

Proof of Proposition 3.5.2. By the previous theorem

$$\int_{\widetilde{\mathcal{M}}} \widetilde{\mu}(\omega + \overline{\omega})^6 = \frac{1}{2} 2^6 \int_{Hilb^3(Kum)} \vartheta(\omega + \overline{\omega})^6$$

and by the description of the Beauville form of $Hilb^3(Kum)$ (see Theorem (4.2.2) of [OG 01])

$$\frac{1}{2} 2^6 \int_{Hilb^3(Kum)} \vartheta(\omega + \overline{\omega})^6 = 2^5 \frac{6!}{3!2^3} \left(\int_{Kum} (\omega_K + \overline{\omega}_K)^2 \right)^3$$

and since the pull back to $\mathcal{J}$ of ω_K is ω

$$2^5 \frac{6!}{3!2^3} \left(\int_{Kum} (\omega_K + \overline{\omega}_K)^2 \right)^3 = 2^5 \frac{6!}{3!2^3} \frac{1}{2^3} \left(\int_{\mathcal{J}} (\omega + \overline{\omega})^2 \right)^3 = 60 \left(\int_{\mathcal{J}} (\omega + \overline{\omega})^2 \right)^3.$$

$\qquad\square$

We now pass to compute other intersection numbers of $\widetilde{\mathcal{M}}$ involving $c_1(\widetilde{\Sigma})$ and $c_1(\widetilde{B})$. Our first target is the following proposition.

Proposition 3.5.7. *Let α_i be cohomology classes in $H^2(\mathcal{J}, \mathbb{Z})$, then*

(37) $$\int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B}) \wedge c_1(\widetilde{\Sigma}) \wedge \widetilde{\mu}(\alpha_1) \wedge \widetilde{\mu}(\alpha_2) \wedge \widetilde{\mu}(\alpha_3) \wedge \widetilde{\mu}(\alpha_4) =$$

$$2^4[(\alpha_1, \alpha_2) \cdot (\alpha_3, \alpha_4) + (\alpha_1, \alpha_3) \cdot (\alpha_2, \alpha_4) + (\alpha_1, \alpha_4) \cdot (\alpha_2, \alpha_3)].$$

Before proving this proposition we fix the notation.

Notation 3.5.8. Let $\mathcal{P}$ be the Poincaré line bundle on $\mathcal{J} \times \widehat{\mathcal{J}}$, let $FM : H^2(\mathcal{J}, \mathbb{Z})$ be the isometry induced by the Fourier-Mukai transform (see [Mu 87]), let Δ and $\overline{\Delta}$ be the diagonal and the anti-diagonal on $\mathcal{J} \times \mathcal{J}$, let I_Δ and $I_{\overline{\Delta}}$ be their respective sheaves of ideals : we can construct on $\mathcal{J} \times \widehat{\mathcal{J}} \times \mathcal{J}$ the flat family of sheaves $p_{1,2}^* \mathcal{P} \otimes p_{1,3}^* I_\Delta \oplus p_{1,2}^* \mathcal{P}^\vee \otimes p_{1,3}^* I_{\overline{\Delta}}$ (here and in the following proposition we denote p_I the projections of $\mathcal{J} \times \widehat{\mathcal{J}} \times \mathcal{J}$, the multiindex I will indicate the image of the projections). Via the identification $\mathcal{J} \leftrightarrow \widehat{\mathcal{J}}$ we can see it as a family of sheaves on $\mathcal{J}$ parametrized by $\mathcal{J} \times \mathcal{J}$.

The next proposition describes the pull backs of the images of the Donaldson's morphism via the modular map $f : \mathcal{J} \times \mathcal{J} \to \mathbf{M}_{(2,0,-2)}^0$ associated to this family.

Proposition 3.5.9. *Let π_i be the projection of $\mathcal{J} \times \mathcal{J}$ to the i-th factor, then*

$$f^* \circ \varphi \circ \mu = -2(\pi_1^* \circ FM + \pi_2^*).$$

Proof. Using the property (19) of the Donaldson's morphism we get

$$f^* \circ \varphi^* \circ \mu(\alpha) = p_{2,3*}(p_1^* \alpha \cup c_2(p_{1,2}^* \mathcal{P} \otimes p_{1,3}^* I_\Delta \oplus p_{1,2}^* \mathcal{P}^\vee \otimes p_{1,3}^* I_{\overline{\Delta}})).$$

Since our family parametrizes sheaves having trivial determinant bundles, and moreover we have $c_1(I_\Delta) = c_1(I_{\overline{\Delta}}) = 0$ and $ch_2(\mathcal{P}) = ch_2(\mathcal{P}^*)$ it follows

(38) $$f^* \circ \varphi^* \circ \mu(\alpha) = -p_{2,3*}(p_1^* \alpha \cup (2ch_2(p_{1,2}^* \mathcal{P}) - ch_2(p_{1,3}^* I_\Delta) - ch_2(p_{1,3}^* I_{\overline{\Delta}}))) =$$

$$-p_{2,3*}(p_1^* \alpha \cup (2p_{1,2}^*(ch_2(\mathcal{P})))) - 2\pi_2^*(\alpha) = -2\pi_1^* \circ FM(\alpha) - 2\pi_2^*(\alpha);$$

the second equality is verified because Δ and $\overline{\Delta}$ act on the 2-cohomology as the identity ($-id^*$ is the identity on H^2). $\qquad\square$

This proposition enables us to prove Proposition 3.5.7.

Proof of Proposition 3.5.7. The modular map f obviously factors through the quotient by the involution $-id$

$$q : \mathcal{J} \times \widehat{\mathcal{J}} \to \mathcal{J} \times \widehat{\mathcal{J}} / - id,$$

its isomorphic image in $\mathbf{M}^0_{(2,0,-2)}$ is just $\Sigma^0_{(2,0,-2)}$.

Recall from (7.3.5) of [OG 03] that $\widetilde{B} \cap \widetilde{\Sigma}$ provide a rational section ψ of the map with general fiber $\mathbb{P}^1$ given by

$$\widetilde{\pi}_{|\widetilde{\Sigma}} : \widetilde{\Sigma} \to \Sigma^0_{(2,0,-2)},$$

the composition $\psi \circ q : \mathcal{J} \times \widehat{\mathcal{J}} \to \widetilde{B} \cap \widetilde{\Sigma}$ is then only a rational map: a resolution $\widetilde{q}$ of $\psi \circ q$ yields the following commutative diagram:

$$
\begin{array}{ccccc}
\widetilde{\mathcal{J} \times \widehat{\mathcal{J}}} & \xrightarrow{\widetilde{q}} & \widetilde{\Sigma} \cap \widetilde{B} & \xrightarrow{\widetilde{i}} & \widetilde{\mathcal{M}} \\
r \downarrow & & \widetilde{\pi}_{|\widetilde{\Sigma}} \downarrow & & \widetilde{\pi} \downarrow \\
\mathcal{J} \times \widehat{\mathcal{J}} & \xrightarrow{q} & \mathcal{J} \times \widehat{\mathcal{J}} / - 1 & \xrightarrow{i} & \mathbf{M}^0_{(2,0,-2)}
\end{array} \quad .
$$

Given $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in H^2(\mathcal{J}, \mathbb{Z})$, since q and $\widetilde{q}$ are generically [2:1], we obtain:

$$\int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B}) \wedge c_1(\widetilde{\Sigma}) \wedge \bigwedge_{i=1}^4 \widetilde{\mu}(\alpha_i) = \int_{\widetilde{B} \cap \widetilde{\Sigma}} \bigwedge_{i=1}^4 \widetilde{i}^* \circ \widetilde{\mu}(\alpha_i) = \frac{1}{2} \int_{\widetilde{\mathcal{J} \times \widehat{\mathcal{J}}}} \bigwedge_{i=1}^4 \widetilde{q}^* \circ \widetilde{i}^* \circ \widetilde{\mu}(\alpha_i).$$

By the commutativity of the previous diagram and since r is birational

$$\frac{1}{2} \int_{\widetilde{\mathcal{J} \times \widehat{\mathcal{J}}}} \bigwedge_{i=1}^4 \widetilde{q}^* \circ \widetilde{i}^* \circ \widetilde{\mu}(\alpha_i) = \frac{1}{2} \int_{\widetilde{\mathcal{J} \times \widehat{\mathcal{J}}}} \bigwedge_{i=1}^4 r^* \circ q^* \circ i^* \circ \varphi^* \circ \mu(\alpha_i) =$$

$$\frac{1}{2} \int_{\mathcal{J} \times \widehat{\mathcal{J}}} \bigwedge_{i=1}^4 q^* \circ i \circ \varphi^* \circ \mu(\alpha_i) = \frac{1}{2} \int_{\mathcal{J} \times \widehat{\mathcal{J}}} \bigwedge_{i=1}^4 f^* \circ \varphi^* \circ \mu(\alpha_i).$$

Proposition 3.5.9 allows us to compute the last term of this equations by means of the intersection form on $H^2(\mathcal{J}, \mathbb{Z})$, we obtain

$$\frac{1}{2} \int_{\mathcal{J} \times \widehat{\mathcal{J}}} \bigwedge_{i=1}^4 f^* \circ \varphi^* \circ \mu(\alpha_i) = \frac{1}{2} \cdot 2^4 \int_{\mathcal{J} \times \widehat{\mathcal{J}}} \bigwedge_{i=1}^4 (\pi_1^* \circ FM(\alpha_i) + \pi_2^* \alpha_i).$$

Since FM is an isometry ([Mu 87]) by an easy computation this last term equals

$$\frac{1}{2} \cdot 2^4 \cdot 2[(\alpha_1, \alpha_2) \cdot (\alpha_3, \alpha_4) + (\alpha_1, \alpha_3) \cdot (\alpha_2, \alpha_4) + (\alpha_1, \alpha_4) \cdot (\alpha_2, \alpha_3)]$$

as asserted in the statement of this proposition. $\qquad\square$

The formula of Proposition 3.5.7 will give information about the Beauville form $B_{\widetilde{\mathcal{M}}}$ of $\widetilde{\mathcal{M}}$ when compared with the Fujiki polarized formula (17) for the same integral. As a first consequence we obtain the following orthogonality relations.

Proposition 3.5.10. *For $\alpha \in H^2(\mathcal{J}, \mathbb{Z})$:*

$$B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{B}), \widetilde{\mu}(\alpha)) = B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{\Sigma}), \widetilde{\mu}(\alpha)) = 0.$$

Proof. Let $\alpha \in H^2(\mathcal{J}, \mathbb{Z})$ be a class such that $(\alpha, \alpha) \neq 0$, then the Proposition 3.5.7 applied in the case $\alpha_i = \alpha$ gives

$$\int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B}) \wedge c_1(\widetilde{\Sigma}) \wedge \widetilde{\mu}(\alpha)^4 = 2^4 \cdot 3(\alpha, \alpha)^2.$$

On the other hand, Fujiki's polarized formula (17) applied in the same case gives

$$\int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B}) \wedge c_1(\widetilde{\Sigma}) \wedge \widetilde{\mu}(\alpha)^4 = a_1 B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{\Sigma}), c_1(\widetilde{B})) B_{\widetilde{\mathcal{M}}}(\widetilde{\mu}(\alpha), \widetilde{\mu}(\alpha))^2 +$$

$$a_2 B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{\Sigma}), \widetilde{\mu}(\alpha)) B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{B}), \widetilde{\mu}(\alpha)) B_{\widetilde{\mathcal{M}}}(\widetilde{\mu}(\alpha), \widetilde{\mu}(\alpha)),$$

(a_1 and a_2 are suitable constants) hence we deduce, under our hypothesis, $B_{\widetilde{\mathcal{M}}}(\widetilde{\mu}(\alpha), \widetilde{\mu}(\alpha)) \neq 0$. Moreover, since $\widetilde{\Sigma}$ and $\widetilde{B}$ are contracted in the Uhlenbeck compactification

$$\int_{\widetilde{\mathcal{M}}} c_1(\widetilde{\Sigma}) \wedge \widetilde{\mu}(\alpha)^5 = \int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B}) \wedge \widetilde{\mu}(\alpha)^5 = 0,$$

and, applying again the Fujiki's polarized formula:

$$B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{\Sigma}), \widetilde{\mu}(\alpha)) B_{\widetilde{\mathcal{M}}}(\widetilde{\mu}(\alpha), \widetilde{\mu}(\alpha))^2 = B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{B}), \widetilde{\mu}(\alpha)) B_{\widetilde{\mathcal{M}}}(\widetilde{\mu}(\alpha), \widetilde{\mu}(\alpha))^2 = 0.$$

The statement of the proposition holds also for the general α since the intersection form of $H^2(\mathcal{J}, \mathbb{Z})$ is non degenerate. $\qquad\square$

It is now possible to determine, up to a scalar factor, the restriction of $B_{\widetilde{\mathcal{M}}}$ to the image of the Donaldson's map.

Proposition 3.5.11. $\exists a \in \mathbb{Q}$ *such that for any* $\alpha \in H^2(\mathcal{J}, \mathbb{Z})$

$$B_{\widetilde{\mathcal{M}}}(\widetilde{\mu}(\alpha), \widetilde{\mu}(\alpha)) = a(\alpha, \alpha).$$

Proof. Applying Proposition 3.5.7 we find

$$(\alpha, \alpha)^2 = \frac{1}{2^4 \cdot 3} \int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B}) \wedge c_1(\widetilde{\Sigma}) \wedge \widetilde{\mu}(\alpha)^4$$

and using the polarized Fujiki's formula (17), thanks to the just proved orthogonality relations:

$$B_{\widetilde{\mathcal{M}}}(\widetilde{\mu}(\alpha), \widetilde{\mu}(\alpha))^2 = \frac{6!}{4! \cdot 6 \cdot c_{\widetilde{\mathcal{M}}} \cdot B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{B}), c_1(\widetilde{\Sigma}))} \int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B}) \wedge c_1(\widetilde{\Sigma}) \wedge \widetilde{\mu}(\alpha)^4$$

and since their squares are proportional, the restriction of $B_{\widetilde{\mathcal{M}}}$ to the image of the Donaldson's morphism and the intersection forms on $\mathcal{J}$ are proportional. $\qquad\square$

We need a few other intersection numbers on $\widetilde{\mathcal{M}}$ to completely determine the Beauville form on $H^2(\widetilde{\mathcal{M}}, \mathbb{Z})$: we collect them in the following proposition.

Proposition 3.5.12. $\forall \omega \in H^0(\Omega^2, \mathcal{J})$ *the followings hold:*

(1) $\int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B}) \wedge c_1(\widetilde{\Sigma}) \wedge \widetilde{\mu}^4(\omega + \overline{\omega}) = 2^4 \cdot 3(\omega + \overline{\omega}, \omega + \overline{\omega})^2,$

(2) $\int_{\widetilde{\mathcal{M}}} c_1(\widetilde{\Sigma})^2 \wedge \widetilde{\mu}^4(\omega + \overline{\omega}) = -2^5 \cdot 3(\omega + \overline{\omega}, \omega + \overline{\omega})^2.$

(3) $\int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B})^2 \wedge \widetilde{\mu}^4(\omega + \overline{\omega}) = -2^4 \cdot 3(\omega + \overline{\omega}, \omega + \overline{\omega})^2,$

Proof. The first formula is again a special case of Proposition 3.5.7. To prove the others consider the more general case of an holomorphic $\mathbb{P}^1$-bundle

$$\rho \overset{i_1}{\hookrightarrow} F$$
$$p \downarrow$$
$$U$$

with a smooth base, non necessarily compact.

Suppose ω be a holomorphic 2-form on F and $Y \subset F$ be an algebraic smooth closed subvariety such that $\rho \cdot Y = d$: then there exists on U a unique holomorphic 2-form ω_U whose pull back to F is ω, and obviously

$$\int_Y (\omega + \overline{\omega})^n = d \int_U (\omega_U + \overline{\omega}_U)^n.$$

Applying this remark to our case, since $\widetilde{\Sigma} \cap \widetilde{B}$ is a rational section of the restriction of $\varphi \circ \widetilde{\pi}$ to $\widetilde{\Sigma}$ (see page 504 of [OG 03]) and by adjunction the normal bundle to $\widetilde{\Sigma}$ has degree -2 on the fibers of such a fibration, we find

$$\int_{\widetilde{\mathcal{M}}} c_1(\widetilde{\Sigma})^2 \wedge \widetilde{\mu}(\omega + \overline{\omega})^4 = -2 \int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B}) \wedge c_1(\widetilde{\Sigma}) \wedge \widetilde{\mu}(\omega + \overline{\omega})^4.$$

Analogously the restriction of $\varphi \circ \widetilde{\pi}$ to $\widetilde{B}$ is still generically a $\mathbb{P}^1$ fibration, $\widetilde{\Sigma}$ intersect its general fiber in 2 points (see [OG 03]) and the degree of the normal bundle of $\widetilde{B}$ on the fiber is -2: we deduce

$$\int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B})^2 \wedge \widetilde{\mu}(\omega + \overline{\omega})^4 = - \int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B}) \wedge c_1(\widetilde{\Sigma}) \wedge \widetilde{\mu}(\omega + \overline{\omega})^4.$$

These two formulas and the first of the statement obviously imply the proposition. $\qquad\square$

We are now ready to prove Theorem 3.5.1.

Proof of 3.5.1. The orthogonality relations have been proved in Proposition 3.5.10. By Fujiki's polarized formula (see (17)) and using Proposition 3.5.12

$$c_{\widetilde{\mathcal{M}}} \frac{4!6}{6!} B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{B}), c_1(\widetilde{\Sigma})) B_{\widetilde{\mathcal{M}}}^2(\widetilde{\mu}(\omega + \overline{\omega}), \widetilde{\mu}(\omega + \overline{\omega})) = \int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B}) \wedge c_1(\widetilde{\Sigma}) \wedge \widetilde{\mu}(\omega + \overline{\omega})^4 =$$
$$2^4 \cdot 3(\omega + \overline{\omega}, \omega + \overline{\omega})^2,$$

$$c_{\widetilde{\mathcal{M}}} \frac{4!6}{6!} B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{\Sigma}), c_1(\widetilde{\Sigma})) B_{\widetilde{\mathcal{M}}}^2(\widetilde{\mu}(\omega + \overline{\omega}), \widetilde{\mu}(\omega + \overline{\omega})) = \int_{\widetilde{\mathcal{M}}} c_1(\widetilde{\Sigma})^2 \wedge \widetilde{\mu}(\omega + \overline{\omega})^4 =$$
$$-2^5 \cdot 3(\omega + \overline{\omega}, \omega + \overline{\omega})^2.$$

$$c_{\widetilde{\mathcal{M}}} \frac{4!6}{6!} B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{B}), c_1(\widetilde{B})) B_{\widetilde{\mathcal{M}}}^2(\widetilde{\mu}(\omega + \overline{\omega}), \widetilde{\mu}(\omega + \overline{\omega})) = \int_{\widetilde{\mathcal{M}}} c_1(\widetilde{B})^2 \wedge \widetilde{\mu}(\omega + \overline{\omega})^4 =$$
$$-2^4 \cdot 3(\omega + \overline{\omega}, \omega + \overline{\omega})^2,$$

Reading only the first and the third terms of these equations, recalling that $c_1(\widetilde{\Sigma}) = 2A$ and using, by Proposition 3.5.11, $B_{\widetilde{\mathcal{M}}}(\widetilde{\mu}(\omega + \overline{\omega}), \widetilde{\mu}(\omega + \overline{\omega})) = a(\omega + \overline{\omega}, \omega + \overline{\omega})$ we find

$$B_{\widetilde{\mathcal{M}}}(A, c_1(\widetilde{B})) = \frac{2^4 \cdot 3}{2a^2 \cdot c_{\widetilde{\mathcal{M}}} \frac{4!6}{6!}} = \frac{2^3 \cdot 3 \cdot 5}{a^2 \cdot c_{\widetilde{\mathcal{M}}}},$$

$$B_{\widetilde{\mathcal{M}}}(A, A) = \frac{-2^5 \cdot 3}{4a^2 \cdot c_{\widetilde{\mathcal{M}}} \frac{4!6}{6!}} = \frac{-2^3 \cdot 3 \cdot 5}{a^2 \cdot c_{\widetilde{\mathcal{M}}}}.$$

$$B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{B}), c_1(\widetilde{B})) = \frac{-2^4 \cdot 3}{a^2 \cdot c_{\widetilde{\mathcal{M}}} \frac{4!6}{6!}} = \frac{-2^4 \cdot 3 \cdot 5}{a^2 \cdot c_{\widetilde{\mathcal{M}}}},$$

Furthermore by Corollary 3.5.2 and Fujiki's formula (15) we get

$$c_{\widetilde{\mathcal{M}}} B_{\widetilde{\mathcal{M}}}^3(\widetilde{\mu}(\omega + \overline{\omega}), \widetilde{\mu}(\omega + \overline{\omega})) = \int_{\widetilde{\mathcal{M}}} \widetilde{\mu}^6(\omega + \overline{\omega}) = 60(\omega + \overline{\omega}, \omega + \overline{\omega})^3$$

which implies $c_{\widetilde{\mathcal{M}}} = \frac{60}{a^3}$ and substituting this in the previous formulas :

$$\begin{pmatrix} B_{\widetilde{\mathcal{M}}}(A,A) & B_{\widetilde{\mathcal{M}}}(A,c_1(\widetilde{B})) \\ B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{B}),A) & B_{\widetilde{\mathcal{M}}}(c_1(\widetilde{B}),c_1(\widetilde{B})) \end{pmatrix} = \begin{pmatrix} -2a & 2a \\ 2a & -4a \end{pmatrix}.$$

The rational number a has then to be integer to respect the integrality of $B_{\widetilde{\mathcal{M}}}$, moreover, since a divides $B_{\widetilde{\mathcal{M}}}$, the primitivity of the Beauville form imposes $|a| = 1$ and, finally, since $c_{\widetilde{\mathcal{M}}} > 0$ then $a = 1$.

The Fujiky constant is then $c_{\widetilde{\mathcal{M}}} = 60$ and the bilinear form $B_{\widetilde{\mathcal{M}}}$ is the one described in the statement of this theorem. $\qquad\square$

By an easy change of base we get:

Corollary 3.5.13. *There exsists an isomorphism of lattices:*

$$(H^2(\widetilde{\mathcal{M}},\mathbb{Z}), B_{\widetilde{\mathcal{M}}}) \simeq (H^2(\mathcal{J},\mathbb{Z}), (,)_{\mathcal{J}}) \oplus_{\perp} \mathbb{Z}A \oplus_{\perp} \mathbb{Z}C$$

where $A \cdot A = C \cdot C = -2$.

References

[AW 97] D.Abramovich, J.Wang, 'Equivariant resolution of singularities in characteristic 0', *Math. Res. Lett.***4** (1997), 427–433.

[BDL 01] J.Bryan, R.Donagi, N.C.Leung, 'G-Bundles on abelian surfaces, hyperkähler manifolds, and stringy Hodge numbers', *Turkish J. Math* **25** (2001), 195–236.

[Be 83] A.Beauville, 'Variété Kählériennes dont la première classe de chern est nulle', *J. Diff. Geom.* **18** (1983).

[Be 99] A.Beauville, 'Counting rational curves on K3 surfaces', *Duke Math. J.* **97** (1999).

[BM 97] E.Bierston, P.Milman, 'Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant', *Invent. Math.* **128** (1997), 207–302.

[Bo 74] F.Bogomolov, 'On the decomposition of Kähler manifolds with trivial canonical class', *Math. USSR-Sb* **22** (1974), 580–583.

[BPV 84] W.Barth, C.Peters, A.Van de Ven, 'Compact complex surfaces', *Springer* (1984)

[De 99] O.Debarre, 'On the Euler characteristic of generalized Kummer varieties', *Amer.J.Math* **121** (1999), 577–586.

[Ei 95] D.Eisenbud, 'Commutative algebra with a view toward algebraic geometry', *Springer GTM* **150** (1995).

[FM 94] R.Friedman, J.Morgan, 'Smooth four-manifolds and complex surfaces', *Ergeb. Math. Grenzgeb. 3. Folge 27 Springer* (1994).

[Fu 87] A.Fujiki, 'On the de Rham Cohomology Group of a Compact Kähler Symplectic Manifold', *Adv. Studies in Pure Math.* **10**(1987) 105–165.

[HL 97] D.Huybrechts, M.Lehn, 'The geometry of moduli spaces of scheaves', *Aspects of Mathematics* **E 31**(1997).

[HM 98] J.Harris, I.Morrison, 'Moduli of curve', *Graduate Texts in Mathematics* **187** (1998).

[Hu 97] D.Huybrechts, 'Birational symplectic manifolds and their deformations', *J. Diff. Geom.* **45** (1997), 488–513.

[Li 93] J.Li, 'Algebraic geometric interpretation of Donaldson's polynomial invariants of algebraic surfaces', *J. Diff. Geom.* **37** (1993), 417–466.

[LP 93] J. Le Potier, 'Systèmes cohérents et structures de niveau', *Astérisque* **214** (1993).

[Ma 99] D.Matsushita, 'On fibre space structures of a projective irreducible symplectic manifolds', *Topology* **38** (1999), 79–83.

[Ma 01] D.Matsushita, Addendum to 'On fibre space structures of a projective irreducible symplectic manifolds', *Topology* **40** (2001), 431–432.

[Mo 93] J. Morgan, 'Comparison of the Donaldson polynomial invariants with their algebro-geometric analogues', *Topology* **32** (1993), 449–488.

[Mu 81] S.Mukai, 'Duality between D(X) and D($\widehat{X}$) with its applications to Picard sheaves', *Nagoya Math. J.* **81** (1981), 153–175.

[Mu 84] S.Mukai, 'Symplectic structure of the moduli space of sheaves on an abelian or K3 surface', *Invent. Math* **77** (1984), 101–116.

[Mu 87] S.Mukai, 'Fourier functor and its application to the moduli of bundles on an abelian variety', *Adv. Stud. Pure Math.* **10** (1987), 515–550.

[NR 69] M.S.Narasimhan, S.Ramanan, 'Moduli of vector bundles on a compact Riemann surface', *Ann. of Math.* **89** (1969), 14–51.

[OG 99] K.O'Grady, 'Desingularized moduli spaces of sheaves on a K3', *J.Reine Angew. Math.* **512** (1999), 49–117.

[OG 01] K.O'Grady, 'Compact Hyperkähler Manifolds', on the web page www.mat.uniroma1.it/people/ogrady.

[OG 03] K.O'Grady, 'A new six dimensional irreducible symplectic variety', *J. Algebraic Geometry* **12** (2003), 435–505.

[Sa 03] J. Sawon, 'Abelian fibred holomorphic symplectic manifolds', *Turkish Journal of Mathematics* **27** (2003), 197–230.

[Yo 01] K.Yoshioka, 'Moduli spaces of stable sheaves on abelian surfaces', *Math.Ann.* **321** (2001), 817–884.

[Yo] K.Yoshioka, 'Irreducibility of moduli spaces of vector bundles on K3 surfaces', *math.AG/9907001*.

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